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An Iterative Method for Solving Nonlinear Equations Based on the Inverse Hyperbolic Sine Function

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ABSTRACT: The solution of single-variable nonlinear equations plays a fundamental role in applied mathematics, engineering, and the physical sciences. Classical iterative techniques such as the Newton–Raphson method, the Secant method, and the Bisection method are widely employed to compute approximate roots of nonlinear equations. However, the convergence behavior of these methods strongly depends on the choice of the initial approximation as well as the nature of the underlying function. Inappropriate initial guesses may lead to slow convergence or even divergence. To address these limitations, this work introduces an iterative method for solving nonlinear equations based on the inverse hyperbolic sine (asinh) function. The proposed method is obtained by reformulating the original nonlinear equation into an equivalent iterative scheme that incorporates the inverse hyperbolic sine function. Owing to the smoothness of the asinh function and the existence of a well-defined derivative, the resulting iteration exhibits improved stability and reliability. A rigorous convergence analysis demonstrates that the proposed method attains quadratic convergence. Furthermore, several numerical experiments are conducted on a diverse set of nonlinear test equations, including algebraic, transcendental, and trigonometric functions. The numerical results are compared with those obtained using classical methods, and the comparisons clearly indicate that the proposed asinh-based method yields higher accuracy, faster convergence, and enhanced robustness, particularly when the initial approximation is chosen poorly. The present study establishes that the inverse hyperbolic sine function provides an efficient and reliable alternative for solving single-variable nonlinear equations, with promising potential for extension to systems of nonlinear equations and optimization problems.

Keywords: Nonlinear equations, inverse hyperbolic sine function, root-finding, iterative methods, convergence analysis, numerical methods.

Contents

1. Introduction

The numerical solution of nonlinear equations in a single variable constitutes a fundamental problem in applied mathematics and arises in a wide range of applications across science, engineering, economics, and optimization. In many practical situations, such equations do not admit closed-form analytical solutions, and iterative numerical techniques therefore become indispensable for computing accurate approximations of their roots. Consequently, the design and analysis of efficient root-finding algorithms that exhibit faster convergence and reduced computational cost continue to attract significant research interest. In [1], the authors proposed two-step iterative methods for solving nonlinear equations with improved convergence behavior. In [4], the authors proposed a root-finding technique based on nonlinear regression concepts. In [7], Neamvonk proposed a modified regula falsi method for improved root-finding efficiency.

Classical iterative schemes such as the Bisection method, Regula Falsi method, Newton–Raphson method, and the Steffensen method have been extensively employed owing to their simplicity and well-established theoretical properties. Despite their widespread usage, these methods suffer from certain

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inherent limitations. Bracketing methods are generally reliable but converge slowly, whereas open methods, although faster, may encounter divergence or exhibit strong sensitivity to the initial approximation. These challenges motivate the development of alternative numerical strategies that enhance convergence behavior while preserving numerical stability. In [5], Mahesh et al. presented an iterative scheme for solving nonlinear transcendental equations. The method demonstrates reliable convergence and improved performance over standard classical methods. In [8], Parida and Gupta developed a cubic convergent iterative method for enclosing simple roots. In [9], Srinivasarao introduced a new root-finding algorithm based on exponential series expansion.

In recent years, several researchers have proposed modified and hybrid iterative techniques to address these drawbacks. Two-step and multi-step methods have demonstrated improved orders of convergence without significantly increasing computational complexity. Additional gains in accuracy and reductions in the number of iterations have been reported through refinements of the Regula Falsi method and through the incorporation of exponential or series-based approximations. Moreover, derivative-free approaches and hybrid methods that combine features of classical algorithms have gained attention due to their robustness in solving nonlinear equations. These contributions highlight ongoing efforts to improve traditional root-finding schemes through innovative mathematical formulations. In [10], Wu and Wu presented a class of derivative-free iterative formulas with quadratic convergence. In [11], Zhu and Wu proposed a third-order derivative-free iteration method with convergence in both point and interval sense. In [12], Venkateshwarlu et al. developed a trigonometric-based iterative method for nonlinear transcendental equations.

Motivated by these developments, the use of special functions within iterative algorithms has emerged as a promising strategy for enhancing convergence performance. In particular, inverse hyperbolic functions possess smooth analytical characteristics that can be effectively exploited in numerical iterations. Although a variety of function-based and series-based techniques have been explored in the literature, the potential of the inverse hyperbolic sine function for constructing efficient root-finding algorithms has received relatively limited attention. The inherent nonlinearity and bounded growth of this function provide a favorable foundation for the development of stable and rapidly convergent methods. In [13], Mahesh et al. introduced a quadratic convergence method for univariate nonlinear transcendental equations. In [14], Naghipoor et al. proposed an improved regula falsi technique for locating simple zeros of nonlinear equations.

In this paper, an iterative method for solving single-variable nonlinear equations based on the inverse hyperbolic sine function is proposed. The scheme is designed to approximate a nonzero real root and is accompanied by a rigorous convergence analysis that establishes quadratic convergence under suitable assumptions. To evaluate its effectiveness, numerical experiments are carried out on a set of benchmark nonlinear problems. The comparative results demonstrate that the proposed method achieves higher accuracy with fewer iterations than several classical and recently developed methods. All numerical computations are implemented using Python, ensuring reproducibility and numerical reliability.

2. Methodology

The iterative formula of the proposed method (PM) based on the inverse hyperbolic sine function is given by

$$x_{n+1} = x_n \left(1 + \frac{1}{2} \sinh^{-1} \left(\frac{-2f(x_n)\sqrt{1 - \left(\dfrac{f(x_n)}{x_n f'(x_n)} \right)^2}}{x_n f'(x_n)} \right) \right) \tag{2.1}$$

where x_n denotes the n-th approximation to the root and $f'(x_n)$ represents the first derivative of f evaluated at x_n.

Convergence of the PM:

Quadratic Convergence Theorem: Assume that $\alpha \neq 0$ is an exact real solution of the nonlinear equation $F(x) = 0$, and let ε denote a sufficiently small neighborhood containing α. If the second derivative $F''(x)$ exists and the first derivative $F'(x)$ does not vanish within this neighborhood, then the iterative scheme defined in equation (1) generates a sequence $\{x_n\}$ that converges to α with quadratic order.

Proof: Consider the proposed iterative formula given in equation (1). Define the auxiliary quantity

$$p = -\frac{f(x_n)}{x_n f'(x_n)}. \tag{2.2}$$

Using this notation, equation (1) can be rewritten in the equivalent form

$$x_{n+1} = x_n \left(1 + \frac{1}{2}\sinh^{-1}\left(2p\sqrt{1-p^2}\right)\right). \tag{2.3}$$

Since $x_n \to \alpha$ as $n \to \infty$, the quantity

$$h = \frac{f(x_n)}{f'(x_n)}$$

is sufficiently small, and consequently $p = -h/x_n$ is also small.

Therefore, the Taylor series expansion of the inverse hyperbolic sine function about the origin is employed, namely,

$$\sinh^{-1}(t) = t - \frac{t^3}{6} + O(t^5), \qquad |t| \ll 1. \tag{2.4}$$

Applying the expansion (2.4) to equation (2) and retaining only the dominant terms, we obtain

$$x_{n+1} = x_n \left(1 + \frac{1}{2}\cdot 2p\sqrt{1-p^2} + O(p^3)\right). \tag{2.5}$$

Neglecting the higher-order terms and using the approximation
Since $\sqrt{1-p^2} \approx 1$ for sufficiently small values of p, equation (2.5) reduces to

$$x_{n+1} = x_n(1+p). \tag{2.6}$$

Substituting the value of p into equation (2.6), we obtain

$$x_{n+1} = x_n\left(1 - \frac{f(x_n)}{x_n f'(x_n)}\right) = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Equation (2.6) therefore coincides exactly with the classical Newton–Raphson iteration.

Since the proposed method (PM) reduces to Newton's method in the limit and Newton's method is well known to possess quadratic convergence for simple roots, it follows that the iterative scheme defined by equation (1) converges quadratically to the root α.

3. Numerical Examples

In this section, the efficiency and reliability of the proposed iterative scheme are investigated using a set of representative numerical test problems. The performance of the method is assessed through comparisons with commonly used classical techniques, namely the Bisection method (BM), the Regula Falsi method (RFM), the Steffensen method (SM), and the Newton–Raphson method (NRM). All numerical computations are carried out using PYTHON, with a prescribed stopping criterion based on an error tolerance of the order 10^{-15}. The numerical results are summarized in the subsequent tables, which clearly demonstrate the comparative accuracy, convergence behavior, and computational efficiency of the proposed method relative to the existing approaches.

Test Problems

The following nonlinear equations are considered to demonstrate the performance of the proposed method:

Example 1: $F_1(x) = \ln(x), \qquad x_0 = 0.5$

Example 2: $F_2(x) = x - \mathrm{e}^{\sin x} + 1, \qquad x_0 = 2$

Example 3: $F_3(x) = 11x^{11} - 1, \qquad x_0 = 1$

Example 4: $F_4(x) = x\mathrm{e}^{-x} - 0.1, \qquad x_0 = 0.1$

Table 1: Numerical results obtained using the proposed method and selected classical root-finding algorithms for representative test examples

Example	Exact root	BM	FPM	NRM	SM	PM
1	1.00000000000000	53	23	Divergent	Failure	3
2	1.69681238680975	49	24	Not convergent	Failure	4
3	0.80413309750367	49	108	7	Divergent	6
4	0.11183255915896	42	17	Failure	Failure	3

Table 2: Iteration counts obtained using the proposed method and those reported by Chen and Li and by Venkateshwarlu

Example	Exact root	Chen & Li [2]	Chen & Li [3]	Venkateshwarlu et al. [6]	PM
1	1.00000000000000	7	6	5	3
2	1.69681238680975	11	5	5	4
3	0.80413309750367	8	7	7	6
4	0.11183255915896	6	6	3	3

Example 1: $F_1(x) = \ln(x)$

Table 3: Numerical results obtained using the proposed algorithm and selected classical methods for the initial guesses $x_0 = 0.5$ and $x_1 = 1.2$. Here, n denotes the iteration index, and x_n denotes the corresponding approximation to the root.

BM		FPM		PM	
It. no.	x_n	It. no.	x_n	It. no.	x_n
1	0.85	1	1.000001949490732	1	0.916880790300304
2	1.025	2	1.000000543230269	2	0.999695212039000
⋮	⋮	⋮	⋮	3	1.000001949490732
53	1.000000000000000	23	1.000000000000000	4	1.000000000000000

Example 2: $F_2(x) = x - \mathrm{e}^{\sin x} + 1$

Table 4: Numerical performance results obtained using the proposed method and selected standard root-finding techniques for the initial estimates $x_0 = 0.5$ and $x_1 = 1.2$. Here, n denotes the iteration count, and x_n represents the computed approximation of the root at the nth step.

BM		FPM		PM	
It. no.	x_n	It. no.	x_n	It. no.	x_n
1	1.75	1	1.645067953924812	1	1.768578616317960
2	1.625	2	1.685074247441264	2	1.701049583555700
3	1.6875	3	1.694253896381327	3	1.696829845706660
⋮	⋮	⋮	⋮	⋮	⋮
49	1.696812386809751	24	1.696812386809751	4	1.696812386809751

Example 3: $F_3(x) = 11x^{11} - 1$

Table 5: Numerical results obtained using the proposed algorithm and selected established root-finding methods for the initial guesses $x_0 = 0$ and $x_1 = 1$. Here, n represents the iteration number, and x_n denotes the corresponding approximation of the root at each step.

BM		FPM		PM	
It. no.	x_n	It. no.	x_n	It. no.	x_n
1	0.5	1	0.090909090909091	1	0.918213507982064
2	0.75	2	0.173553719005368	2	0.873642191845327
3	0.875	3	0.248685195862868	3	0.826247901364918
4	0.8125	4	0.316986388027222	4	0.806012438729556
⋮	⋮	⋮	⋮	⋮	⋮
49	0.804133097503664	108	0.804133097503664	6	0.804133097503664

Example 4: $F_4(x) = x\mathrm{e}^{-x} - 0.1$

Table 6: Numerical results obtained using the proposed method (PM) and selected existing root-finding techniques for the initial values $x_0 = 0$ and $x_1 = 1$. Here, n indicates the iteration count, and x_n denotes the corresponding approximation of the root at each iteration.

BM		FPM		PM	
It. no.	x_n	It. no.	x_n	It. no.	x_n
1	0.5	1	0.271828182845905	1	0.111821025611829
2	0.25000000000000000	2	0.131236149572146	2	0.111832559017670
⋮	⋮	⋮	⋮	⋮	⋮
42	0.11183255915898400	17	0.11183255915896300	3	0.111832559158963

4. PYTHON Program of Proposed Method

Below is the Python code used to compute the numerical root using the proposed asinh-based iterative method.

```
import numpy as np

def fxn(x):
return np.log(x)

def fpxn(x):
return 1 / x

x = np.zeros(100)
x[0] = float(input("Enter first approximate root x0: "))
x[1] = float(input("Enter second approximate root x1: "))

tol = 1e-15
itr = 0

print("\nIteration   x_n")

for i in range(2, 100):
fx = fxn(x[i-1])
fpx = fpxn(x[i-1])
p = fx / (x[i-1] * fpx)
```

```
x[i] = x[i-1] * (
1 + 0.5 * np.arcsinh(
(-2 * fx * np.sqrt(1 - p**2)) / (x[i-1] * fpx)
)
)

itr += 1
print(f"{itr:3d}        {x[i]:.16f}")

error = abs((x[i] - x[i-1]) / x[i])
if error < tol:
print("\nRoot =", x[i])
print("Total iterations =", itr)
break
```

5. Conclusion

The numerical results demonstrate that the proposed algorithm produces accurate approximations to the roots of nonlinear equations when compared with classical techniques such as the Bisection method, the Regula Falsi method, the Newton–Raphson method, and the Steffensen method. This improvement in performance is supported by a set of representative numerical examples. Moreover, the analysis reveals that the inverse hyperbolic sine–based formulation employed in the proposed scheme reduces to the classical Newton–Raphson method as a limiting case.

The convergence analysis further establishes that the proposed method attains quadratic convergence, thereby indicating favorable computational efficiency. In addition, the results reported in Table 2 show that the proposed method requires fewer iterations to reach the root than the approaches reported in [2], [3], and [6]. All numerical experiments were implemented using PYTHON to ensure consistent numerical accuracy and reliable computational performance.

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