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Structural and Topological Properties of Multiplicative Normed Linear Spaces

Imtiyaz M. D. and B. Krishna Reddy

ABSTRACT: The primary objective of this research is to formulate a robust topological structure for Multiplicative Normed Linear Spaces (MNLS). By re-examining fundamental analytical concepts through a multiplicative lens, we establish Key theoretical contributions include a characterization of multiplicative closed sets via limit points and the derivation of a multiplicative version of the Cantor Intersection Principle. We further demonstrate that finite-dimensional MNLS are inherently complete and that the Heine-Borel property holds, meaning a subspace is compact if and only if it is multiplicatively closed and bounded. Additionally, the study confirms that continuous mappings on compact domains are necessarily uniformly continuous. These results provide the essential theoretical scaffolding required for future developments in non-Newtonian calculus and fixed-point theory.

Keywords: Multiplicative boundedness and total boundedness, multiplicative compactness and precompactness, multiplicative continuity and uniform continuity.

Contents

1. Introduction

Conventional mathematical analysis is predicated largely on additive arithmetic. Yet, numerous physical and biological processes—such as bacterial proliferation, financial compounding, and disparate growth phenomena—are more naturally modeled through ratios rather than differences. To address this, Bashirov et al. [1] pioneered the framework of multiplicative calculus, which substitutes the additive difference operator with a multiplicative one.

This non-Newtonian paradigm has since been expanded from metric spaces to vector spaces. Recent scholarship has formalized the concept of a Multiplicative Normed Linear Space (MNLS) to establish a geometric setting compatible with multiplicative calculus [4, 5]. Although foundational algebraic properties have been documented, a comprehensive topological treatment—specifically regarding completeness, the structure of finite-dimensional spaces, and compact embeddings—has remained underexplored.

This study aims to fill this theoretical void by constructing a rigorous topological framework for MNLS. We concentrate on the space $(\Omega, \|\cdot\|^*)$, where the norm adheres to multiplicative algebraic axioms. The primary contributions of this work are:

- We derive a multiplicative version of the Cantor Intersection Theorem.
- We characterize finite-dimensional MNLS, establishing that every such space is inherently multiplicatively complete.

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- We prove the Multiplicative Heine-Borel Theorem, demonstrating that compactness in finite dimensions is equivalent to being multiplicatively closed and bounded.
- We prove the preservation of these properties under continuous mappings, specifically proving that multiplicative continuity on a compact set implies uniform continuity.

These results solidify the structural basis required for future applications in fixed-point theory and non-Newtonian differential equations.

2. Preliminaries

In this section, we revisit the core concepts of multiplicative calculus. To distinguish our framework from standard additive texts and minimize confusion, we denote the underlying linear space by Ω.
Definition 2.1 [1](Multiplicative Metric Space). Consider a non-empty set Ω. A function $\rho : \Omega \times \Omega \to [1, \infty)$ is classified as a multiplicative metric if it satisfies the following axioms for all elements $u, v, w \in \Omega$:

1. $\rho(u, v) \geq 1$, with equality holding if and only if $u = v$.
2. $\rho(u, v) = \rho(v, u)$ (Symmetry).
3. $\rho(u, w) \leq \rho(u, v) \cdot \rho(v, w)$ (Multiplicative Triangle Inequality).

The pair (Ω, ρ) constitutes a multiplicative metric space.
Definition 2.2 [5](Multiplicative Normed Linear Space). Let Ω be a linear space over the field $\mathbb{R}$. A mapping $\|\cdot\|^* : \Omega \to [1, \infty)$ is termed a multiplicative norm if, for all vectors $u, v \in \Omega$ and scalars $\alpha \in \mathbb{R}$, the conditions below are met:

1. $\|u\|^* \geq 1$ and $\|u\|^* = 1 \iff u = 0$ (Non-degeneracy).
2. $\|\alpha u\|^* = (\|u\|^*)^{|\alpha|}$ (Multiplicative Homogeneity).
3. $\|u + v\|^* \leq \|u\|^* \cdot \|v\|^*$ (Multiplicative Triangle Inequality).

The pair $(\Omega, \|\cdot\|^*)$ is defined as a Multiplicative Normed Linear Space (MNLS). This structure naturally induces a metric topology where the distance is given by $\rho(u, v) = \|u - v\|^*$.
Definition 2.3[5] Let $(\Omega, \|\cdot\|^*)$ be an MNLS. We define the fundamental topological structures as follows:

1. **Multiplicative Open Ball:** For any center $u \in \Omega$ and radius $r > 1$, the open ball is the set:
$$B(u, r) = \{v \in \Omega : \|u - v\|^* < r\}.$$
2. **Multiplicative Open Set:** A subset $O \subseteq \Omega$ is deemed multiplicative open if every point $u \in O$ is the center of some ball $B(u, r)$ entirely contained within O.
3. **Multiplicative Closed Set:** A subset $C \subseteq \Omega$ is multiplicative closed provided its complement, $\Omega \setminus C$, is multiplicative open.
4. **Multiplicative Closure:** The closure of a set A, denoted $\overline{A}$, is defined as the intersection of all multiplicative closed sets containing A. Equivalently, $u \in \overline{A}$ if and only if $B(u, r) \cap A \neq \emptyset$ for all $r > 1$.

Definition 2.4 [5] Let $(\Omega, \|\cdot\|^*)$ be an MNLS. We characterize a sequence $\{u_n\}_{n=1}^{\infty} \subset \Omega$ as **multiplicative convergent** to a limit $u \in \Omega$ provided that, for any arbitrary tolerance $\varepsilon > 1$, there exists an integer $N \in \mathbb{N}$ satisfying:
$$\|u_n - u\|^* < \varepsilon, \quad \forall n \geq N.$$
This convergence is symbolically represented as $u_n \to^* u$ as $n \to \infty$, or equivalently, $\|u_n - u\|^* \to 1$.

Example 2.5 [5]: Consider the real space $(\mathbb{R}, \|\cdot\|^*)$ equipped with the exponential norm $\|v\|^* = e^{|v|}$. For the harmonic sequence $(u_n) = \left(\frac{1}{n}\right)$, the limit is the zero vector 0, since $\|u_n - 0\|^* = e^{|1/n|} \to e^0 = 1$ as $n \to \infty$. Thus, $u_n \to^* 0$.

Definition 2.6 [5] A sequence $\{u_n\}_{n=1}^{\infty}$ within Ω is designated as a **multiplicative Cauchy sequence** if the multiplicative distance between terms eventually vanishes. Formally, for every $\varepsilon > 1$, one can find an index $N \in \mathbb{N}$ such that:

$$\|u_n - u_m\|^* < \varepsilon, \quad \forall n, m \geq N.$$

Example 2.7: In the aforementioned space $(\mathbb{R}, e^{|\cdot|})$, take the sequence $(u_n) = \left(\frac{1}{n}\right)$. We observe that:

$$\|u_n - u_m\|^* = e^{|u_n - u_m|} = e^{\left|\frac{1}{n} - \frac{1}{m}\right|} \to 1 \quad \text{as} \quad m, n \to \infty.$$

This confirms the sequence is Cauchy in the multiplicative sense.

3. Topological Foundations

In this section, we systematically reconstruct the fundamental topological concepts of analysis within the framework of MNLS.

Theorem 3.1: A subset K of an MNLS is multiplicative closed if and only if it contains all its multiplicative limit points. That is, $K = \overline{K}$.

Proof: Let K be closed. If u is a limit point of K but $u \notin K$, then $u \in K^c$. Since K^c is open, there exists a ball $B(u, r) \subset K^c$, which implies $B(u, r) \cap K = \emptyset$. This contradicts that u is a limit point. Thus K contains all limit points.

Conversely, assume K contains all its limit points. Let $v \in K^c$. If every ball around v intersected K, v would be a limit point and thus in K (by hypothesis), which is a contradiction. Thus, there exists a ball around v disjoint from K, proving K^c is open and K is closed.

Definition 3.2:A subset K within a Multiplicative Normed Linear Space $(\Omega, \|\cdot\|^*)$ is designated as **multiplicative precompact** provided that its multiplicative closure, $\overline{K}$, is multiplicative compact.

Example 3.3: Consider the space $(\mathbb{R}, \|\cdot\|^*)$ equipped with the exponential norm $\|u\|^* = e^{|u|}$. Take the open interval $K = (0, 1) \subseteq \mathbb{R}$. We observe that the closure of this set corresponds to the closed interval $\overline{K} = [0, 1]$. Since $[0, 1]$ is multiplicative compact, it follows that the original set K satisfies the condition for precompactness.

We now establish the relationship between convergent sequences and their subsequences.

Theorem 3.4: Consider an MNLS $(\Omega, \|\cdot\|^*)$. Suppose a sequence $\{u_n\} \subset \Omega$ converges multiplicatively to a limit u. Then, every subsequence $\{u_{n_k}\}$ extracted from $\{u_n\}$ must also converge multiplicatively to the same limit u.

Proof: Fix an arbitrary tolerance $\varepsilon > 1$. The hypothesis that $u_n \to^* u$ guarantees the existence of a threshold integer $N \in \mathbb{N}$ such that the inequality $\|u_n - u\|^* < \varepsilon$ holds for every index $n \geq N$.

Recall that for any subsequence, the indices are strictly increasing, which implies $n_k \geq k$ for all k. Consequently, whenever we select $k \geq N$, it necessarily follows that $n_k \geq N$. This ensures that the terms of the subsequence satisfy:

$$\|u_{n_k} - u\|^* < \varepsilon.$$

.Therefore, we conclude that $u_{n_k} \to^* u$ as $k \to \infty$.

The following theorem provides a necessary and sufficient condition for the convergence of Cauchy sequences.

Theorem 3.5: Let $(\Omega, \|\cdot\|^*)$ be an MNLS and let $\{u_n\}$ be a multiplicative Cauchy sequence. The sequence $\{u_n\}$ is multiplicative convergent if and only if it contains a convergent subsequence.

Proof: $(\Rightarrow)$ If $\{u_n\}$ converges to a limit u, then by the previous result (Theorem 3.4), every subsequence must essentially converge to the same limit u.

$(\Leftarrow)$ Conversely, suppose $\{u_n\}$ is a multiplicative Cauchy sequence possessing a subsequence $\{u_{n_k}\}$ that converges to a point $u \in \Omega$. We demonstrate that the entire sequence converges to u. Select an arbitrary $\varepsilon > 1$. By the definition of a Cauchy sequence, we can find an integer $N_1 \in \mathbb{N}$ such that the distance between terms is controlled:

$$\|u_n - u_m\|^* < \sqrt{\varepsilon}, \quad \forall n, m \geq N_1.$$

Additionally, since the subsequence $u_{n_k} \to^* u$, there exists an integer $N_2 \in \mathbb{N}$ satisfying:

$$\|u_{n_k} - u\|^* < \sqrt{\varepsilon}, \quad \forall k \geq N_2.$$

Define $N = \max\{N_1, N_2\}$. For any index $n \geq N$, we can select a specific index k sufficiently large such that $n_k \geq N$. Applying the multiplicative triangle inequality yields:

$$\|u_n - u\|^* \leq \|u_n - u_{n_k}\|^* \cdot \|u_{n_k} - u\|^* < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Consequently, we conclude that $u_n \to^* u$.

Theorem 3.6: Let $(\Omega, \|\cdot\|^*)$ be an MNLS. If a subspace $K \subseteq \Omega$ is multiplicative precompact, then it is necessarily multiplicative totally bounded.

Proof: Assume that K is multiplicative precompact. By definition, this implies that its multiplicative closure, $\overline{K}$, is multiplicative compact. We invoke the standard topological property that every compact metric space is totally bounded; thus, $\overline{K}$ is multiplicative totally bounded.

Fix an arbitrary tolerance $\varepsilon > 1$. Since $\overline{K}$ is totally bounded, there exists a finite set of points $\{u_1, u_2, \ldots, u_n\} \subseteq \overline{K}$ that acts as a $\sqrt{\varepsilon}$-net. Specifically, for any element $u \in \overline{K}$, we can find an index $i \in \{1, \ldots, n\}$ such that:

$$\|u_i - u\|^* < \sqrt{\varepsilon}.$$

Next, we approximate these centers with points strictly inside K. Since each u_i belongs to the closure $\overline{K}$, for every i, there exists a corresponding point $u_i' \in K$ sufficiently close to u_i such that $\|u_i - u_i'\|^* < \sqrt{\varepsilon}$.

We now demonstrate that $\{u_1', \ldots, u_n'\}$ forms an ε-net for K. Consider any arbitrary point $u \in K$. By the triangle inequality and our construction:

$$\|u - u_i'\|^* \leq \|u - u_i\|^* \cdot \|u_i - u_i'\|^* < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Since u was arbitrary, the finite set $\{u_1', \ldots, u_n'\} \subseteq K$ covers K within a radius of ε. Consequently, K is multiplicative totally bounded. The converse need not be true. Let $(\mathbb{Q}, \|\cdot\|^*)$ be a MNLS with $\|u\|^* = e^{|u|}$. Consider the set

$$K = \{q \in \mathbb{Q} : 0 \leq q < \sqrt{3}\}.$$

Then K is multiplicative totally bounded (as a subset of the bounded interval $[0, \sqrt{3}]$), but it is not multiplicative precompact because its closure in $\mathbb{Q}$ is not compact (it misses the limit point $\sqrt{3} \notin \mathbb{Q}$.

One of the central results of this study is the multiplicative analogue of Cantor's Intersection Theorem.

Definition 3.7: Let D be a non-empty, bounded subset of a MNLS. The **diameter** of D is defined as

$$\operatorname{diam}(D) = \sup\{\|u - v\|^* : u, v \in D\}.$$

Example 3.8: Let $(\mathbb{R}, \|\cdot\|^*)$ be a MNLS with $\|u\|^* = e^{|u|}$. The diameter of $[0, 1]$ is e.

Theorem 3.9:[Multiplicative Cantor Intersection Theorem] A Multiplicative Normed Linear Space (MNLS) $(\Omega, \|\cdot\|^*)$ is multiplicatively complete if and only if every nested sequence $\{D_n\}$ of non-empty, multiplicatively closed, and bounded subsets of Ω with $\lim_{n\to\infty} \operatorname{diam}(D_n) = 1$ has a singleton intersection. **Proof:** $(\Rightarrow)$ Assume Ω is complete. For a sequence $\{D_n\}$ satisfying the hypothesis, select $u_n \in D_n$. Since $D_m \subseteq D_n$ for $m \geq n$, it follows that $\|u_n - u_m\|^* \leq \operatorname{diam}(D_n)$. As $\operatorname{diam}(D_n) \to 1$, $\{u_n\}$ is a multiplicative Cauchy sequence converging to some $u \in \Omega$. Since the tail $\{u_k\}_{k\geq n}$ lies in the closed set D_n, we have $u \in \bigcap_{n=1}^{\infty} D_n$. Uniqueness is guaranteed because if $v \in \bigcap D_n$, then $\|u - v\|^* \leq \operatorname{diam}(D_n) \to 1$, implying $u = v$.

$(\Leftarrow)$ Conversely, let $\{u_n\}$ be a multiplicative Cauchy sequence. Define D_n as the multiplicative closure of the tail $\{u_k : k \geq n\}$. These sets are nested, closed, and non-empty. Since $\{u_n\}$ is Cauchy, $\operatorname{diam}(D_n) \to 1$. By hypothesis, $\bigcap_{n=1}^{\infty} D_n = \{u\}$. Since u resides in the closure of every tail, $u_n \to^* u$, proving comp

Theorem 3.10: If a subspace K of a multiplicative complete MNLS $(\Omega, \|\cdot\|)$ is multiplicative totally bounded, then it is multiplicative pre-compact.

Proof: Let K be multiplicative totally bounded and let $(p_n) \subset K$ be an arbitrary sequence. By total boundedness, K can be covered by finitely many multiplicative balls of radius $r_k = 1 + \frac{1}{k}$ for $k \in \mathbb{N}$.

We proceed inductively.For $k = 1$, cover K with balls of radius $r_1 = 2$. we can find , at least one ball contains a subsequence of (p_n); let p_{n_1} be a term in this subsequence. For general k, we construct a nested sequence of balls $B_k(z_k, 1+\frac{1}{k})$ intersecting the infinite tail of the previous subsequence. Let $D_k = \bigcap_{j=1}^{k} \overline{B_j}$.The sets D_k are multiplicative closed, nested, and satisfy $\text{diam}(D_k) \leq (1+\frac{1}{k})^2$. Since Ω is complete and $\text{diam}(D_k) \to 1$ as $k \to \infty$, the Cantor Intersection Theorem implies $\bigcap_{k=1}^{\infty} D_k = \{z\}$ for a unique $z \in \Omega$. Since $p_{n_k} \in D_k$, we have $\|p_{n_k} - z\| \leq \text{diam}(D_k) \to 1$. Thus, $\lim p_{n_k} = z$, proving K is multiplicative pre-compact.

Theorem 3.11: Let $A \subset (\Omega, \|\cdot\|)$ be a subset of a Multiplicative Normed Linear Space. Then A is multiplicatively totally bounded if and only if every sequence in A admits a multiplicative Cauchy subsequence.

Proof:($\Rightarrow$) Assume A is multiplicatively totally bounded. For each $k \in \mathbb{N}$, A can be covered by finitely many balls of radius $1+\frac{1}{k}$. By the Pigeonhole Principle, we can inductively extract a subsequence (p_{n_k}) such that for any fixed k, the infinite tail of the subsequence resides within a single ball of radius $1+\frac{1}{k}$. For any $m, n \geq k$, the multiplicative triangle inequality yields:

$$\|p_{n_m} - p_{n_n}\| \leq \left(1+\frac{1}{k}\right)^2.$$

Since $(1+\frac{1}{k})^2 \to 1$ as $k \to \infty$, (p_{n_k}) is a multiplicative Cauchy sequence.

($\Leftarrow$) Conversely, suppose A is not totally bounded. Then there exists $\epsilon > 1$ such that no finite union of ϵ-balls covers A. We construct a sequence (p_n) inductively: select $p_1 \in A$ and choose $p_{m+1} \in A \setminus \bigcup_{j=1}^{m} B(p_j, \epsilon)$. This construction ensures $\|p_i - p_j\| \geq \epsilon$ for all distinct i, j, implying (p_n) possesses no multiplicative Cauchy subsequence.

4. Multiplicative Completion

In this section, we address the problem of completing an arbitrary MNLS. Let $(\Omega, \|\cdot\|^*)$ be a MNLS. We consider the set of all multiplicative Cauchy sequences in Ω, denoted by $\mathcal{C}(\Omega)$.

Definition 4.1:[Equivalence of Cauchy Sequences] Two multiplicative Cauchy sequences $\{u_n\}$ and $\{v_n\}$ in Ω are said to be **equivalent**, denoted by $\{u_n\} \sim \{v_n\}$, if

$$\lim_{n\to\infty} \|u_n - v_n\|^* = 1.$$

We define the completion space $\hat{\Omega}$ as the set of equivalence classes, i.e., $\hat{\Omega} = \mathcal{C}(\Omega)/\sim$. We denote the equivalence class of a sequence $\{u_n\}$ by $\hat{u} = [\{u_n\}]$.

The mapping $\|\cdot\|_{\hat{\Omega}} : \hat{\Omega} \to [1, \infty)$ defined by

$$\|\hat{u} - \hat{v}\|_{\hat{\Omega}} = \lim_{n\to\infty} \|u_n - v_n\|^*$$

is a well-defined multiplicative metric on $\hat{\Omega}$.

Theorem 4.2:[Existence and Uniqueness] Let $(\Omega, \|\cdot\|^*)$ be a MNLS. Then there exists a multiplicatively complete MNLS $(\hat{\Omega}, \|\cdot\|_{\hat{\Omega}})$ and a multiplicative isometry $\phi : \Omega \to \hat{\Omega}$ such that $\phi(\Omega)$ is multiplicative dense in $\hat{\Omega}$. Furthermore, the space $\hat{\Omega}$ is unique up to multiplicative isometry.

Proof: We define the embedding $\phi : \Omega \to \hat{\Omega}$ by mapping each $u \in \Omega$ to the equivalence class of the constant sequence $(u, u, \dots)$. For any $u, v \in \Omega$, $\|\phi(u) - \phi(v)\|_{\hat{\Omega}} = \|u - v\|^*$, proving ϕ is an isometry. Density follows because any Cauchy sequence $\{u_n\}$ is the limit of the sequence of equivalence classes corresponding to its terms. Completeness is shown via a standard diagonal argument.

5. Completeness and Compactness in Finite Dimensional MNLS

We now turn to the most significant topological results for applications multiplicative completeness and the multiplicativeHeine-Borel behaviour of finite-dimensional spaces.

Theorem 5.1:[Completeness] Every finite-dimensional MNLS $(\Omega, \|\cdot\|^*)$ is multiplicatively complete.

Proof: Let $\dim(\Omega) = n$ with a basis $\{\xi_1, \dots, \xi_n\}$. Consider a multiplicative Cauchy sequence $\{u_m\}$ in Ω, expressed as $u_m = \sum_{i=1}^{n} c_i^{(m)} \xi_i$. The Cauchy condition on $\{u_m\}$ implies that each scalar coefficient sequence $\{c_i^{(m)}\}_m$ is a Cauchy sequence in $\mathbb{R}$. The completeness of $\mathbb{R}$ ensures the existence of limits $c_i^{(m)} \to c_i$. Consequently, the vector $u = \sum_{i=1}^{n} c_i \xi_i$ is the unique multiplicative limit of $\{u_m\}$, establishing the completeness of Ω.

Definition 5.2 [5]:[Multiplicative Compactness] A subspace $K \subseteq \Omega$ is **multiplicative compact** if every sequence in K admits a multiplicative convergent subsequence converging to a point within K.

Theorem 5.3:[Multiplicative Heine-Borel Theorem] Let $(\Omega, \|\cdot\|^*)$ be a finite-dimensional MNLS. A subspace $K \subseteq \Omega$ is multiplicative compact if and only if it is multiplicative closed and multiplicative bounded.

Proof:($\Rightarrow$) If K is multiplicative compact, it must be totally bounded (hence bounded) and complete (hence closed), consistent with standard metric space theory.

($\Leftarrow$) Conversely, assume K is multiplicative closed and bounded. Let $\{u_m\}$ be an arbitrary sequence in K. Given a fixed basis for Ω, the multiplicative boundedness of K implies that the sequence of scalar coefficients for $\{u_m\}$ is bounded in $\mathbb{R}^n$. By the classical Bolzano-Weierstrass theorem, there exists a subsequence $\{u_{m_k}\}$ for which the scalar coefficients converge. This ensures $\{u_{m_k}\}$ converges multiplicatively to a limit u. Since K is multiplicative closed, we conclude $u \in K$.

6. Multiplicative Continuity

In this section, we investigate the preservation of topological properties under multiplicative mappings. We focus on the relationship between compactness and uniform continuity, establishing a multiplicative analogue of the Heine-Cantor Theorem.

Definition 6.1:[Multiplicative Uniform Continuity] Let $(S_1, \|\cdot\|_1^*)$ and $(S_2, \|\cdot\|_2^*)$ be MNLS. A function $f : S_1 \to S_2$ is said to be multiplicative uniformly continuous if for every $\epsilon > 1$, there exists $\delta > 1$ such that

$$\|g(u) - g(v)\|_2^* < \epsilon \quad \text{whenever} \quad \|u - v\|_1^* < \delta$$

for all $u, v \in S_1$.

Theorem 6.2: Let K be a multiplicative compact subspace of an MNLS $(S, \|\cdot\|_1^*)$. If $g : K \to S_2$ is multiplicative continuous, then g is multiplicative uniformly continuous on K.

Proof: Suppose, for the sake of contradiction, that g is not multiplicative uniformly continuous on K. Then there exists some $\epsilon > 1$ such that for every $\delta > 1$, there exist points $u, v \in K$ with $\|u - v\|_1^* < \delta$ but $\|g(u) - g(v)\|_2^* \geq \epsilon$. For each $n \in \mathbb{N}$, let $\delta_n = 1 + \frac{1}{n}$. We can choose a pair of sequences $\{u_n\}$ and $\{v_n\}$ in K such that:

$$\|u_n - v_n\|_1^* < 1 + \frac{1}{n} \quad \text{and} \quad \|g(u_n) - g(v_n)\|_2^* \geq \epsilon.$$

Since K is multiplicative compact, the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that converges multiplicatively to some limit $z \in K$, i.e., $\|u_{n_k} - z\|_1^* \to 1$. We now show that the corresponding subsequence $\{v_{n_k}\}$ also converges to z. By the multiplicative triangle inequality:

$$\|v_{n_k} - z\|_1^* \leq \|v_{n_k} - u_{n_k}\|_1^* \cdot \|u_{n_k} - z\|_1^*.$$

As $k \to \infty$, $\|u_{n_k} - z\|_1^* \to 1$ (by convergence) and $\|v_{n_k} - u_{n_k}\|_1^* \to 1$ (by construction). Thus, $\|v_{n_k} - z\|_1^* \to 1$, implying $v_{n_k} \to^* z$.Since f is multiplicative continuous at z, we have $g(u_{n_k}) \to^* g(z)$ and $g(v_{n_k}) \to^* g(z)$. Consequently,

$$\|g(u_{n_k}) - g(v_{n_k})\|_2^* \leq \|g(u_{n_k}) - g(z)\|_2^* \cdot \|g(z) - g(v_{n_k})\|_2^* \to 1 \cdot 1 = 1.$$

This implies that for sufficiently large k, $\|g(u_{n_k}) - g(v_{n_k})\|_2^* < \epsilon$, which contradicts the assumption that $\|g(u_n) - g(v_n)\|_2^* \geq \epsilon$ for all n. Therefore, g must be multiplicative uniformly continuous.

Theorem 6.3:[Extension Principle] Let S_3 be a dense subspace of an MNLS $(S_1, \|\cdot\|_1^*)$, and let $(S_2, \|\cdot\|_2^*)$ be a multiplicative complete MNLS. If $g : S_3 \to S_2$ is multiplicative uniformly continuous, then there exists a unique multiplicative continuous extension $\bar{g} : S_1 \to S_2$.

Proof: Let $u \in S_1$. Since S_3 is dense, there exists a sequence $\{u_n\} \subset S_3$ such that $u_n \to^* u$. Since $\{u_n\}$ is convergent, it is a multiplicative Cauchy sequence. By the uniform continuity of f, the image sequence $\{f(u_n)\}$ is a multiplicative Cauchy sequence in S_2. Since S_2 is complete, this sequence converges to a limit $v \in S_2$. We define $\bar{f}(u) = v$. Standard arguments show that this definition is independent of the choice of sequence $\{u_n\}$ and that the resulting extension $\bar{f}$ preserves uniform continuity.

7. Conclusion

In this paper, we have provided a comprehensive topological framework for Multiplicative Normed Linear Spaces. By systematically reconstructing the foundations of analysis, we have demonstrated that the multiplicative structure allows for a rich theory parallel to classical functional analysis. Our main contributions include the characterization of multiplicative closure, the Multiplicative Cantor's Intersection Theorem, and the proof that finite-dimensional MNLS are inherently complete. These findings provide the necessary structural basis for further advancements in non-Newtonian calculus.

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References

1. A. E. Bashirov, E. M. Kurpınar, and A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (2008), no. 1, 36–48.
2. S. Lang, *Real and Functional Analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 142, Springer-Verlag, New York, 1993.
3. E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, Inc., New York, 1990.
4. M. Özavşar and A. C. Çevikel, *Fixed point of multiplicative contraction mappings on multiplicative metric space*, arXiv preprint arXiv:1205.5131 (2012).
5. R. Chugh, A. Nandal, and N. Kumar, *Multiplicative normed linear space and its topological properties*, Indian J. Sci. Technol. **10** (2017), no. 10, 1–9.
6. M. T. Nair, *Functional Analysis: A First Course*, PHI Learning Private Limited, Delhi, 2014.
7. Badshah-e-Rome and M. Sarwar, *Characterization of multiplicative metric completeness*, Int. J. Anal. Appl. **10** (2016), no. 2, 90–94.

Imtiyaz M. D.,
Assistant Professor, Department of Mathematics,
MVS Government Arts and Science College(A),
Mahabubnagar Dist., Palamuru University.
E-mail address: imtiyazmd83@gmail.com

and

B. Krishna Reddy,
Professor, Department of Mathematics,
University College of Science,
Osmania University, Hyderabad, India.
E-mail address: bkreddy345@osmania.ac.in