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Upper Bound Estimates of Toeplitz Determinant of Fourth Order for Certain Class of Close-to-Convex Harmonic Mappings

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ABSTRACT: In this present work, we obtain an upper bound of fourth order Topelitz determinant for a function in the class of Harmonic close to convex functions. The upper bounds on $|T_{4,1}(f)|$ found for $f \in \mathbb{M}(\alpha, 1, 1)$ and $|T_{2,2}(f)|, |a_3^2 - a_2a_4|, |a_4^2 - a_3a_5|$ and $|T_{4,2}(f)|$ where $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$ were obtained.

Keywords: Univalent functions, harmonic univalent function, close to convex function, coefficient inequalities, fourth order Topelitz determinants.

Contents

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $\mathbb{U} \subset \mathbb{C}$ if both u and v are real harmonic in $\mathbb{U}$, with series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n = h + \bar{g}. \tag{1.1}$$

Here $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$, $(\mid b_1 \mid < 1; z \in \mathbb{U})$, are analytic functions in $\mathbb{U}$. We call h the analytic part and g the co-analytic part of f. Let $\mathbb{H}$ be the class of all complex-valued harmonic functions that are normalized on the unit disk $\mathbb{U}$, that is
$\mathbb{H} = \{f : \mathbb{U} \to \mathbb{C}/f \text{ is harmonic, with } f(0) = a_0 = 0, f_z(0) = a_1 = 1\}$.
Since the Jacobian of f is given by $\mid h' \mid^2 - \mid g' \mid^2$, by Lewy's *et al* [8], it is locally univalent and sense-preserving if and only if $\mid g' \mid < \mid h' \mid$, or equivalently, the dilatation $\omega = g'/h'$ with $h'(z) \neq 0$ has the property $\mid \omega \mid < 1$ in $\mathbb{U}$. The subclass of $\mathbb{H}$ that is univalent and sense-preserving in $\mathbb{U}$ is denoted by $\mathbb{S}_H$.
The family $\mathbb{S}$ of analytic univalent and normalized functions in $\mathbb{U}$ is a subclass of $\mathbb{S}_H$ with $g(z) \equiv 0$. The family of all functions $f \in \mathbb{S}_H$ with the additional condition of normalization that $f_{\bar{z}}(0) = b_1 = 0$, is denoted by $\mathbb{S}_H^0$. That is
$\mathbb{S}_H^0 = \{f \in \mathbb{S}_H / f_{\bar{z}}(0) = b_1 = 0\}$. If $f = h + \overline{g} \in \mathbb{S}_H^0$ then

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n \qquad (|b_1| = 0; z \in \mathbb{U})$$

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Let $\mathbb{A}$ represent the family of analytic functions f, normalized by $f(0) = 0$ and $f'(0) = 1$, defined on the open unit disk $\mathbb{U}$ on the complex plane $\mathbb{C}$ and $\mathbb{K}(\alpha)$ denotes the class of functions $f \in \mathbb{A}$ such that

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad \left(-\frac{1}{2} \leq \alpha < 1; z \in \mathbb{U}\right). \tag{1.2}$$

In particular, the elements in $\mathbb{K}$(-1/2) are close-to-convex but are not necessarily starlike in $\mathbb{U}$, for $0 \leq \alpha < 1$, the elements in $\mathbb{K}(\alpha)$ are known to be convex functions of order α in $\mathbb{U}$.

A domain Ψ is said to be close-to-convex in $\mathbb{C}$, if it can be expressed as a union of non-intersecting half-lines. According to the results of Kaplan [7], an analytic function f is called close-to-convex if there exists a univalent convex analytic function ϕ defined in $\mathbb{U}$ such that

$$Re\left(\frac{f'(z)}{\phi'(z)}\right) > 0, \qquad z \in \mathbb{U}.$$

Moreover, a planar harmonic mapping $f : \mathbb{U} \to \mathbb{C}$ is close-to-convex if it is one-to-one function and $f(\mathbb{U})$ is a close-to-convex domain.The class of such close-to-convex harmonic mappings is denoted by $\mathbb{C}_H^0$.

Definition 1.1 *A harmonic mapping $f = h + \bar{g}$ is said to be in the class $\mathbb{M}(\alpha, \zeta, n)$ if $h \in \mathbb{K}(\alpha)$, for some $\alpha \in [-\frac{1}{2}, 1)$, given by (1.2) and g satisfies the condition.*

$$g'(z) = \zeta z^n h'(z) \qquad \left(\zeta \in \mathbb{C} \quad with \mid \zeta \mid \leq \frac{1}{2n-1}; n \in \mathbb{N} = 1, 2, 3, \ldots.\right).$$

For n = 1, α = -1/2 and $\mid \zeta \mid = 1$, the class $\mathbb{M}$(-1/2, ζ, 1) was introduced by Bharanedhar and Ponnusamy*et al.* [5]. In 2019, Sun *et al.* [12], investigated upper bounds of the third Hankel determinats for the class $\mathbb{M}(\alpha)$ of close-to-convex harmonic mappings.

Definition 1.2 *A harmonic mapping $f = h + \bar{g}$ of the form (1.1) is said to be in the class $\mathbb{M}(\alpha, 1, 1)$ if $h \in \mathbb{K}(\alpha)$, for some $\alpha \in [-\frac{1}{2}, 1)$, g satisfies the given condition*

$$g'(z) = zh'(z) \quad \left(z \in \mathbb{U}\right). \tag{1.3}$$

Definition 1.3 *A harmonic mapping $f = h + \bar{g}$ of the form (1.1) is said to be in the class $\mathbb{M}(\alpha, \frac{1}{3}, 2)$ if $h \in \mathbb{K}(\alpha)$, for some $\alpha \in [-\frac{1}{2}, 1)$, g satisfies the condition*

$$g'(z) = \frac{1}{3} z^2 h'(z) \quad (z \in \mathbb{U}). \tag{1.4}$$

The q^{th} symmetric Toeplitz determinant $T_{q,n}(f)$ for $f \in \mathcal{A}$ is defined as follows:

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_n & \ldots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \ldots & a_n \end{vmatrix} \tag{1.5}$$

where $q \geq 2$, $n \geq 1$, and $a_1 = 1$ (refer to [2]).

2. Literature Review

The study of Toeplitz determinants associated with the members of $\mathbb{S}$ play an important part in the research area of complex analysis. Ahuja *et al.* [1], Ali *et al.* [2], Arif *et al.* [3], Wang *et al.* [13], Zhang *et al.* [19] have investigated on the symmetric Toeplitz determinant $\mid \mathbb{T}_q(n) \mid$ for subclasses of $\mathbb{S}$ with small values of n and q. Yakaiah *et al.* [16], [18], [17] worked on fourth Hankel and Toeplitz determinant for certain univalent analytic functions subordinate to cosz, 1+sinz and inverse of reciprocal of bounded turning functions. Rao and Sharma [10] estimated an upper bounds of fourth Hankel and Toeplitz determinant for starlike and convex functions associated with Nephroid domain.

Xiao Yuan Wang *et al.* [14] estimated an upper bound of $|T_{2,2}(h)|, |T_{3,1}(h)|, |T_{3,1}(g)|, |T_{3,2}(h)|, |T_{3,2}(g)|$ for $f \in \mathbb{M}(\alpha, \zeta, n)$.

1. Wang *et al.* [14] shown that

$$\begin{aligned}|T_{2,2}(h)| \leq &\frac{2}{9}(1-\alpha)^2(2\alpha^2-6\alpha+9)\\ |T_{2,2}(g)| \leq &\frac{1}{18}\end{aligned}$$

   for $f \in \mathbb{M}(\alpha, \zeta, 2)$

2. If $f \in \mathbb{M}(\alpha, \zeta, 1)$,

$$|T_{3,1}(h)| \leq \begin{cases} \frac{1}{9}\left(8\alpha^4-34\alpha^3+71\alpha^2-72\alpha+36\right), & -\frac{1}{2} \leq \alpha \leq \frac{1}{2},\\ \frac{1}{9}\left(-2\alpha^3+25\alpha^2-44\alpha+30\right), & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

   and

$$|T_{3,1}(g)| \leq \frac{1}{3}(1-\alpha)$$

3. If $f \in \mathbb{M}(\alpha, \zeta, 2)$,

$$|T_{3,2}(h)| \leq \begin{cases} \frac{1}{108}(1-\alpha)^3(2\alpha^2-7\alpha)\left(10\alpha^2-27\alpha+36\right), & -\frac{1}{2} \leq \alpha \leq \frac{1}{7},\\ \frac{5}{108}(1-\alpha)^3(2\alpha^2-7\alpha)\left(2\alpha^2-4\alpha+7\right), & \frac{1}{7} \leq \alpha < 1. \end{cases}$$

   and

$$\mid T_{3,2}(g) \mid = \mid 2b_3^2b_4 \mid \leq \frac{1}{243}(1-\alpha)$$

4. Yakaiah *et al.* [15] shown that

$$|T_{4,1}(h)| \leq \frac{77}{9} \quad for \;\; h \in \mathbb{K} \tag{2.1}$$

3. Motivation and the Proposed Research Problem

Many researchers have worked upon computing upper bounds to $|T_{4,1}(f)|$ and $|T_{4,2}(f)|$ for $f \in \mathbb{A}$ (for instance, [18], [17], [10]). However, there is no such analogues results conceding harmonic univalent functions. This motivates us to compute $|T_{4,1}(f)|$ for $f \in \mathbb{M}(\alpha, 1, 1)$ and $|T_{4,2}(f)|$ for $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$. Motivated by the work of Xiao- Yuan Wang *et al.* [14] and Sun *et al.* [12], in this paper, upper bounds of fourth Toplitz determinants for the classes $\mathbb{M}(\alpha, 1, 1)$, $\mathbb{M}(\alpha, \frac{1}{3}, 2)$ for $h \in \mathbb{K}(\alpha)$, for some $\alpha \in$ [-1/2,1), given by (1.2) and g that satisfies the condition by (1.3) and (1.4) respectively.

3.1. Techniques used

In doing our research work we made use of the following Lemmas.
Let $\mathcal{P}$ be the class of all functions with positive real part with series representation.

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \text{ for } z \in \mathbb{U}. \tag{3.1}$$

Unless otherwise stated throughout this paper, we assume the series representation of $p \in \mathcal{P}$ is of the form (3.1).

Lemma 3.1 *If $p \in \mathbb{P}$ and p is of form (3.1), then the sharp estimates*

$$|c_r| \leq 2, \tag{3.2}$$

$$|c_{r+s} - \mu c_r c_s| \leq 2, \qquad for \;\; 0 \leq \mu \leq 1, \tag{3.3}$$

$$|c_r c_s - c_t c_u| \leq 4, \qquad for \;\; r+s = t+u. \tag{3.4}$$

hold for each r, s, t, $u \in \mathbb{N} = \{1, 2, 3, ...\}$. Inequalities (3.2), and (3.3) are proved in [11], [6], respectively. Inequality (3.4) is obvious.

Lemma 3.2 *[9] Let $p \in \mathbb{P}$ and $\mu \in \mathbb{C}$. Then $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.*

Lemma 3.3 *[4] Let $p \in \mathbb{P}$. Then for complex numbers J, K, L*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2\Big[|J| + |K - 2J| + |J - K + L|\Big] \tag{3.5}$$

4. Main Results

Let $f \in \mathbb{M}(\alpha, 1, 1)$ be of the form (1.1). Then there exists $p(z) \in \mathcal{P}$ of the form (3.1) such that

$$p(z) = \frac{1}{1-\alpha}\left(1 + \frac{zh''(z)}{h'(z)} - \alpha\right) \qquad \left(-\frac{1}{2} \leq \alpha < 1, z \in \mathbb{U}\right). \tag{4.1}$$

It follows that

$$n(n-1)a_n = (1-\alpha)\sum_{k=1}^{n-1} ka_kc_{n-k} \quad (n \geq 2). \tag{4.2}$$

Equating the coefficients of similar powers of z, we obtain

$$a_2 = \frac{1}{2}(1-\alpha)c_1, \tag{4.3}$$

$$a_3 = \frac{1}{6}(1-\alpha)\big[(1-\alpha)c_1^2 + c_2\big], \tag{4.4}$$

$$a_4 = \frac{1}{24}(1-\alpha)\big[(1-\alpha)^2c_1^3 + 3(1-\alpha)c_1c_2 + 2c_3\big], \tag{4.5}$$

$$a_5 = \frac{1}{120}(1-\alpha)[(1-\alpha)^3c_1^4 + 6(1-\alpha)^2c_1^2c_2 + 8(1-\alpha)c_1c_3 + 3(1-\alpha)c_2^2 + 6c_4]. \tag{4.6}$$

By the power series representations of h and g for $f \in \mathbb{M}(\alpha, 1, 1)$, we see that

$$b_1 = 0, \quad (k+1)b_{k+1} = ka_k \quad (k \geq 1, a_1 = 1).$$

That yields

$$b_2 = \frac{1}{2}, \tag{4.7}$$

$$b_3 = \frac{1}{3}(1-\alpha)c_1, \tag{4.8}$$

$$b_4 = \frac{1}{8}(1-\alpha)\big[(1-\alpha)c_1^2 + c_2\big], \tag{4.9}$$

$$b_5 = \frac{1}{30}(1-\alpha)[(1-\alpha)^2c_1^3 + 3(1-\alpha)c_1c_2 + 2c_3]. \tag{4.10}$$

Using the power series representations of h and g associated with $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$, we see that

$$d_1 = 0, d_2 = 0, \quad (k+2)d_{k+2} = \frac{1}{3}ka_k \quad (k \geq 1, a_1 = 1).$$

That yields

$$d_3 = \frac{1}{9}, \tag{4.11}$$

$$d_4 = \frac{1}{12}(1-\alpha)c_1, \tag{4.12}$$

$$d_5 = 1/30(1-\alpha)\big[(1-\alpha)c_1^2 + c_2\big]. \tag{4.13}$$

4.1. Upper bound estimates of $|T_{4,1}(h)|$ and $|T_{4,1}(g)|$ where $f \in \mathbb{M}(\alpha, 1, 1)$

Theorem 4.1 *If* $f = h + \bar{g} \in \mathbb{M}(\alpha, 1, 1)$, *then*

$$|T_{4,1}(h)| \leq \begin{cases} 1 + \dfrac{1}{648}(1-\alpha)^2\left[\alpha^6 - 16\alpha^5 + 581\alpha^4 - 2472\alpha^3 + 5788\alpha^2 - 6540\alpha + 6668\right], & -\dfrac{1}{2} \leq \alpha < 0, \\ 1 + \dfrac{1}{648}(1-\alpha)^2\left[-4\alpha^6 + 16\alpha^5 - 8\alpha^4 - 594\alpha^3 + 3673\alpha^2 - 5588\alpha + 4611\right], & 0 \leq \alpha < 1. \end{cases} \tag{4.14}$$

and

$$|T_{4,1}(g)| \leq \begin{cases} \dfrac{1}{5184}\left[5380\alpha^4 - 35596\alpha^3 + 88341\alpha^2 - 75194\alpha + 41101\right], & -\dfrac{1}{2} \leq \alpha < -\dfrac{2}{7}, \\ \dfrac{1}{576}\left[576\alpha^4 - 3840\alpha^3 + 9601\alpha^2 - 10406\alpha + 4537\right], & -\dfrac{2}{7} \leq \alpha < 1. \end{cases} \tag{4.15}$$

Proof: As $f \in \mathbb{M}(\alpha, 1, 1)$, by the definition of $T_{4,1}(h)$

$$T_{4,1}(h) = (1 - a_2^2)^2 - (a_2a_3 - a_4)^2 + (a_3^2 - a_2a_4)^2 - (a_2 - a_2a_3)^2 + 2(a_2^2 - a_3)(a_3 - a_2a_4)$$

By using the values of a_2, a_3, a_4 from (4.3), (4.4), (4.5) we get

$$\begin{aligned} T_{4,1}(h) =& 1 + \frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^6c_1^8 - 2(1-\alpha)^5c_1^6c_2 - 12(1-\alpha)^4c_1^5c_3 + 9(1-\alpha)^4c_1^4c_2^2 - 252(1-\alpha)^4 \\ & c_1^6 - 8(1-\alpha)^3c_1^2c_2^3 - 288(1-\alpha)^3c_1^4c_2 + 12(1-\alpha)^3c_1^3c_2c_3 + 36(1-\alpha)^3c_1^2c_3^2 + 252(1-\alpha)^2c_1^2c_2^2 \\ & - 48(1-\alpha)^2c_1c_2^2c_3 + 3600(1-\alpha)^2c_1^4 + 16(1-\alpha)^2c_2^4 + 144(1-\alpha)c_1c_2c_3 + 1152(1-\alpha)c_1^2c_2 \\ & - 15552c_1^2 - 1152c_2^2 - 144c_3^2\Big] \end{aligned}$$

If $\left(-\frac{1}{2} \leq \alpha < 0\right)$

$$\begin{aligned} T_{4,1}(h) =& 1 + \frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^6c_1^8 - 2(1-\alpha)^5c_1^6c_2 - 6(1-\alpha)^4c_1^5c_3 + 4(1-\alpha)^5c_1^4c_2^2 - 8(1-\alpha)^3c_1^2c_2^3 \\ & - 24(1-\alpha)^2c_1c_2^2c_3 - 288(1-\alpha)^3c_1c_2^4 + 252(1-\alpha)^2c_1^2c_2^2 + 36(1-\alpha)c_1c_2c_3 - 6(1-\alpha)^4c_1^5c_3 \\ & + 12(1-\alpha)^3c_1^3c_2c_3 + 36(1-\alpha)^2c_1^2c_3 - 144c_3^2 + 96(1-\alpha)c_1c_2c_3 - 1152c_2^2 + 768(1-\alpha)c_1^2c_2 \\ & + 16(1-\alpha)^2c_2^4 - 24(1-\alpha)^2c_1c_2^2c_3 + 3600(1-\alpha)^2c_1^4 + 384(1-\alpha)c_1^2c_2 + 12(1-\alpha)c_1c_2c_3 \\ & - 252(1-\alpha)^4c_1^6 - 15552c_1^2 + 5(1-\alpha)^5c_1^4c_2^2\Big] \end{aligned}$$

$$\begin{aligned} T_{4,1}(h) =& 1 + \frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^4c_1^5\big[(1-\alpha)^2c_1^3 - 2(1-\alpha)c_1c_2 - 6c_3\big] + 4(1-\alpha)^2c_1c_2^2\big[(1-\alpha)^2c_1^3 \\ & - 2(1-\alpha)c_1c_2 - 6c_3\big] - 12(1-\alpha)c_1c_2\big[24(1-\alpha)^2c_1^3 - 21(1-\alpha)c_1c_2 - 3c_3\big] - 6(1-\alpha)^2c_1^2c_3 \\ & \big[(1-\alpha)^2c_1^3 - 2(1-\alpha)c_1c_2 - 6c_3\big] - 144c_3\big[c_3 - \frac{2}{3}(1-\alpha)c_1c_2\big] - 15552c_1^2 - 1152c_2\big[c_2 \\ & - \frac{2}{3}(1-\alpha)c_1^2\big] - 16(1-\alpha)^2c_2^2\big[c_1c_3 - c_2^2\big] - 8(1-\alpha)^2c_1c_2^2c_3 + 3600(1-\alpha)^2c_1^4 + 384(1-\alpha)c_1^2c_2 \\ & + 12(1-\alpha)c_1c_2c_3 - 252(1-\alpha)^4c_1^6 + 5(1-\alpha)^5c_1^4c_2^2\Big] \end{aligned}$$

$$\begin{aligned} T_{4,1}(h) =& 1 + \frac{1}{20736}(1-\alpha)^2\Big[\big[(1-\alpha)^2c_1^3 - 2(1-\alpha)c_1c_2 - 6c_3\big]\big[(1-\alpha)^4c_1^5 - 4(1-\alpha)^2c_1^2c_3 + 4(1-\alpha)^2c_1 \\ & c_2^2 - 2(1-\alpha)^2c_1^2c_3\big] - 12(1-\alpha)c_1c_2\big[24(1-\alpha)^2c_1^3 - 21(1-\alpha)c_1c_2 - 3c_3\big] - 144c_3\big[c_3 - \frac{2}{3}(1-\alpha) \\ & c_1c_2\big] - 1152c_2\big[c_2 - \frac{2}{3}(1-\alpha)c_1^2\big] - 16(1-\alpha)^2c_2^2\big[c_1c_3 - c_2^2\big] - 8(1-\alpha)^2c_1c_2^2c_3 + 3600(1-\alpha)^2c_1^4 \\ & + 384(1-\alpha)c_1^2c_2 + 12(1-\alpha)c_1c_2c_3 - 252(1-\alpha)^4c_1^6 + 5(1-\alpha)^5c_1^4c_2^2 - 15552c_1^2\Big] \end{aligned}$$

By applying the triangle inequality and using Lemmas 3.1, 3.2 and 3.3, we obtain the bound for $|T_{4,1}(h)|$

$$\begin{aligned}\Big|T_{4,1}(h)\Big| \leq & 1+\frac{1}{20736}(1-\alpha)^2\Big[\big[4(\alpha^2-2\alpha+4)\big]\big[32(1-\alpha)^4+32(1-\alpha)^2+16(1-\alpha)^2\big]+12(1-\alpha)(4)\\ & \big[6(32\alpha^2-50\alpha+17)\big]+144(2)(2)+1152(2)(2)+16(1-\alpha)^2(4)(4)+(8)(16)(1-\alpha)^2+3600\\ & (16)(1-\alpha)^2+384(1-\alpha)(8)+12(1-\alpha)(8)+252(1-\alpha)^4(64)+5(1-\alpha)^5(64)+15552(4)\Big]\end{aligned}$$

$$\Big|T_{4,1}(h)\Big| \leq 1+\frac{1}{648}(1-\alpha)^2\Big[\alpha^6-16\alpha^5+581\alpha^4-2472\alpha^3+5788\alpha^2-6540\alpha+6668\Big]\ \Big(-\frac{1}{2}\leq\alpha<0\Big) \tag{4.16}$$

If $\Big(0\leq\alpha<1\Big)$

$$\begin{aligned}T_{4,1}(h) =& 1+\frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^6c_1^8-2(1-\alpha)^5c_1^6c_2-6(1-\alpha)^4c_1^5c_3-6(1-\alpha)^4c_1^5c_3+4(1-\alpha)^4c_1^4c_2^2\\ & +5(1-\alpha)^4c_1^4c_2^2-252(1-\alpha)^4c_1^6-8(1-\alpha)^3c_1^2c_2^3-288(1-\alpha)^3c_1^4c_2+12(1-\alpha)^3c_1^3c_2c_3+36\\ & (1-\alpha)^3c_1^2c_3^2+252(1-\alpha)^2c_1^2c_2^2-24(1-\alpha)^2c_1c_2^2c_3-24(1-\alpha)^2c_1c_2^2c_3+3600(1-\alpha)^2c_1^4+16\\ & (1-\alpha)^2c_2^4+144(1-\alpha)c_1c_2c_3+1152(1-\alpha)c_1^2c_2-15552c_1^2-1152c_2^2-144c_3^2\Big]\\ =& 1+\frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^6c_1^8-2(1-\alpha)^5c_1^6c_2-6(1-\alpha)^4c_1^5c_3+6(1-\alpha)^4c_1^4c_2^2-12(1-\alpha)^3c_1^2\\ & c_2^3-36(1-\alpha)^2c_1c_2^2c_3+3(1-\alpha)^4c_1^4c_2^2+4(1-\alpha)^3c_1^2c_2^3-6(1-\alpha)^4c_1^5c_3+12(1-\alpha)^3c_1^3c_2c_3+36\\ & (1-\alpha)^2c_1^2c_3^2-144c_3^2+144(1-\alpha)c_1c_2c_3-1152c_2^2+1152(1-\alpha)c_1^2c_2+3600(1-\alpha)^2c_1^4-12\\ & (1-\alpha)^2c_1c_2^2c_3+16(1-\alpha)^2c_2^4-15552c_1^2-252(1-\alpha)^4c_1^6+252(1-\alpha)^2c_1^2c_2^2-288(1-\alpha)^3c_1^4c_2\Big]\\ =& 1+\frac{1}{20736}(1-\alpha)^2\Big[(1-\alpha)^4c_1^5\big[(1-\alpha)^2c_1^3-2(1-\alpha)c_1c_2-6c_3\big]+6(1-\alpha)^2c_1c_2^2\big[(1-\alpha)^2c_1^3\\ & -2(1-\alpha)c_1c_2-6c_3\big]+3(1-\alpha)^4c_1^4c_2^2+4(1-\alpha)^3c_1^2c_2^3-6(1-\alpha)^2c_1^2c_3\big[(1-\alpha)^2c_1^3-2(1-\alpha)c_1\\ & c_2-6c_3\big]-144c_3\big[c_3-(1-\alpha)c_1c_2\big]-1152c_2\big[c_2-(1-\alpha)c_1^2\big]+3600(1-\alpha)^2c_1^4-12(1-\alpha)^2c_1\\ & c_2^2c_3+16(1-\alpha)^2c_2^4-15552c_1^2-288(1-\alpha)^3c_1^4c_2-252(1-\alpha)^4c_1^6+252(1-\alpha)^2c_1^2c_2^2\Big]\\ =& 1+\frac{1}{20736}(1-\alpha)^2\Big[\big[(1-\alpha)^2c_1^3-2(1-\alpha)c_1c_2-6c_3\big]\big[(1-\alpha)^4c_1^5+6(1-\alpha)^2c_1\big[c_2^2-c_1c_3\big]\big]\\ & +3(1-\alpha)^2c_1c_2^2\big[(1-\alpha)^2c_1^3+\frac{4}{3}(1-\alpha)c_1c_2-4c_3\big]-144c_3\big[c_3-(1-\alpha)c_1c_2\big]+\big[c_2-(1-\alpha)\\ & c_1^2\big]\big[252(1-\alpha)^3c_1^4+252(1-\alpha)^2c_1^2c_2-1152c_2\big]+3600(1-\alpha)^2c_1^4-15552c_1^2+16(1-\alpha)^2c_2^2\\ & -288(1-\alpha)^3c_1^4c_2\Big]\end{aligned}$$

By applying the triangle inequality and using Lemmas 3.1, 3.2 and 3.3, we obtain the bound for

$$\begin{aligned}\Big|T_{4,1}(h)\Big| \leq & 1+\frac{1}{20736}(1-\alpha)^2\Big[\big[4(4-\alpha)^2\big]\big[(1-\alpha)^4(32)+6(1-\alpha)^2(2)(2)\big]+3(1-\alpha)^2(8)\big[4(\alpha^2-2\alpha+3\big]\\ & +144(2)(2)+(2)\big[252(1-\alpha)^3 16+252(1-\alpha)^2(8)+1152(2)\big]+3600(1-\alpha)^2(16)+15552(4)\\ & +16(1-\alpha)^2 4+288(1-\alpha)^3(32)\Big]\end{aligned}$$

$$\Big|T_{4,1}(h)\Big| \leq 1+\frac{1}{648}(1-\alpha)^2\Big[4\alpha^6+16\alpha^5-8\alpha^4-594\alpha^3+3673\alpha^2-5588\alpha+4611\Big]\qquad\Big(0\leq\alpha<1\Big) \tag{4.17}$$

By equations (4.16), (4.17) that

$$|T_{4,1}(h)| \leq \begin{cases} 1+\frac{1}{648}(1-\alpha)^2\Big[\alpha^6-16\alpha^5+581\alpha^4-2472\alpha^3+5788\alpha^2-6540\alpha+6668\Big] & \Big(-\frac{1}{2}\leq\alpha<0\Big) \\ 1+\frac{1}{648}(1-\alpha)^2\Big[-4\alpha^6+16\alpha^5-8\alpha^4-594\alpha^3+3673\alpha^2-5588\alpha+4611\Big] & \Big(0\leq\alpha<1\Big) \end{cases} \tag{4.18}$$

As $f \in \mathbb{M}(\alpha,1,1)$, the following function expansion $T_{4,1}(g)$
$T_{4,1}(g) = (1-b_2^2)^2-(b_2b_3-b_4)^2+(b_3^2-b_2b_4)^2-(b_2-b_2b_3)^2+2(b_2^2-b_3)(b_3-b_2b_4)$
By using the values of b_2, b_3, b_4 from (4.7), (4.8), (4.9) that

$$\begin{aligned}
(1-b_2^2) =& \frac{3}{4} \\
(b_2b_3-b_4) =& \frac{1}{24}(1-\alpha)\Big[-3(1-\alpha)c_1^2+4c_1-3c_2\Big] \\
(b_3^2-b_2b_4) =& \frac{1}{144}(1-\alpha)\Big[7(1-\alpha)c_1^2-9c_2\Big] \\
(b_2-b_2b_3) =& \frac{1}{6}\Big[-(1-\alpha)c_1+3\Big] \\
(b_2^2-b_3)) =& \frac{1}{12}\Big[-4(1-\alpha)c_1+3\Big] \\
(b_3-b_2b_4) =& \frac{1}{48}(1-\alpha)\Big[-3(1-\alpha)c_1^2-3c_2+16c_1\Big]
\end{aligned}$$

By applying the triangle inequality and using Lemmas 3.1, 3.3 and 3.2, we obtain the bound for the $(1-b_2^2), (b_2b_3-b_4), (b_3^2-b_2b_4), (b_2-b_2b_3), (b_2^2-b_3)$ and $(b_3-b_2b_4)$.

$$\begin{cases}
|1-b_2^2| \leq \frac{3}{4} \quad \Big(-\frac{1}{2}\leq\alpha<1\Big) \\
|b_2b_3-b_4| \leq \frac{1}{6}(1-\alpha)\Big[-6\alpha+13\Big] \quad \Big(-\frac{1}{2}\leq\alpha<1\Big) \\
|b_3^2-b_2b_4| \leq \begin{cases} \frac{1}{72}(1-\alpha)\Big[-14\alpha+23\Big] \quad \Big(-\frac{1}{2}\leq\alpha<-\frac{2}{7}\Big) \\ \frac{1}{8}(1-\alpha)\Big] \quad \Big(-\frac{2}{7}\leq\alpha<1\Big) \end{cases} \\
|(b_2-b_2b_3| \leq \frac{1}{6}\Big[-2\alpha+5\Big] \quad \Big(-\frac{1}{2}\leq\alpha<1\Big) \\
|b_2^2-b_3| \leq \frac{1}{12}\Big[-8\alpha+11\Big] \quad \Big(-\frac{1}{2}\leq\alpha<1\Big) \\
|b_3-b_2b_4| \leq \frac{1}{24}(1-\alpha)\Big[-6\alpha+25\Big] \quad \Big(-\frac{1}{2}\leq\alpha<1\Big)
\end{cases} \tag{4.19}$$

$$\Big|T_{4,1}(g)\Big| \leq |1-b_2^2|^2-|b_2b_3-b_4|^2+|b_3^2-b_2b_4|^2-|b_2-b_2b_3|^2+2|b_2^2-b_3||b_3-b_2b_4| \tag{4.20}$$

By using the bounds of $|1-b_2^2|, |b_2b_3-b_4|, |b_3^2-b_2b_4|, |b_2-b_2b_3|, |b_2^2-b_3|, |b_3-b_2b_4|$ from (4.19) that
If $\Big(-\frac{1}{2}\leq\alpha<-\frac{2}{7}\Big)$

$$\Big|T_{4,1}(g)\Big| \leq \left|\frac{3}{4}\right|^2+\left|\frac{1}{6}(1-\alpha)\Big[-6\alpha+13\Big]\right|^2+\left|\frac{1}{72}(1-\alpha)\Big[-14\alpha+23\Big]\right|^2+\left|\frac{1}{6}\Big[-2\alpha+5\Big]\right|^2+2\left|\frac{1}{18}\right.$$

$$\left.(1-\alpha)^2\Big[\alpha^2-2\alpha+4\Big]\right|\left|\frac{1}{24}(1-\alpha)\Big[-6\alpha+25\Big]\right|$$

$$\begin{aligned}
\leq& \frac{9}{16}+\frac{1}{36}(1-\alpha)^2\Big[36\alpha^2-156\alpha+169\Big]+\frac{1}{5184}(1-\alpha)^2\Big[196\alpha^2-644\alpha+529\Big]+\frac{1}{36}\Big[4\alpha^2-20\alpha+25\Big] \\
&+\frac{1}{144}(1-\alpha)\Big[48\alpha^2-266\alpha+275\Big]
\end{aligned}$$

$$\Big|T_{4,1}(g)\Big| \leq \frac{1}{5184}\Big[5380\alpha^4-35596\alpha^3+88341\alpha^2-75194\alpha+41101\Big] \qquad \Big(-\frac{1}{2}\leq\alpha<-\frac{2}{7}\Big) \tag{4.21}$$

If $\Big(-\frac{2}{7}\leq\alpha<1\Big)$

$$\begin{aligned}\Big|T_{4,1}(g)\Big| \leq & \Big[\frac{3}{4}\Big]^2+\Big[\frac{1}{6}(1-\alpha)\Big[-6\alpha+13\Big]\Big]^2+\Big[\frac{1}{8}(1-\alpha)\Big]^2+\Big[\frac{1}{6}\Big[-2\alpha+5\Big]\Big]^2+2\Big[\frac{1}{18}(1-\alpha)^2\Big[\alpha^2-2\alpha\\ & +4\Big]\Big]\Big[\frac{1}{24}(1-\alpha)\Big[-6\alpha+25\Big]\Big]\\ \leq & \frac{9}{16}+\frac{1}{36}(1-\alpha)^2\Big[36\alpha^2-156\alpha+169\Big]+\frac{1}{64}(1-\alpha)^2+\frac{1}{36}\Big[4\alpha^2-20\alpha+25\Big]+\frac{1}{144}(1-\alpha)\\ & \Big[48\alpha^2-266\alpha+275\Big]\end{aligned}$$

$$\Big|T_{4,1}(g)\Big| \leq \frac{1}{576}\Big[576\alpha^4-3840\alpha^3+9601\alpha^2-10406\alpha+4537\Big] \qquad \Big(-\frac{2}{7}\leq\alpha<1\Big) \tag{4.22}$$

By equation (4.21) and (4.22) that

$$|T_{4,1}(g)| \leq \begin{cases}\dfrac{1}{5184}\left[5380\alpha^4-35596\alpha^3+88341\alpha^2-75194\alpha+41101\right], & -\dfrac{1}{2}\leq\alpha<-\dfrac{2}{7},\\ \dfrac{1}{576}\left[576\alpha^4-3840\alpha^3+9601\alpha^2-10406\alpha+4537\right], & -\dfrac{2}{7}\leq\alpha<1.\end{cases}$$

□

Remark 4.1 *For $\alpha = 0$ in Theorem 4.1, our results improves the existing bounds obtained by Yakaiah et al. [15]*

4.2. Upper bound estimates of $|T_{2,2}(f)|, |a_3^2-a_2a_4|, |a_4^2-a_3a_5|$ and $|T_{4,2}(f)|$ where $f\in\mathbb{M}(\alpha,\frac{1}{3},2)$

Theorem 4.2 *If $f=h+\bar{g}\in\mathbb{M}(\alpha,\frac{1}{3},2)$, then*

$$|T_{2,2}(h)| \leq \frac{2}{9}(1-\alpha)^2\Big[2\alpha^2-6\alpha+9\Big] \qquad \Big(-\frac{1}{2}\leq\alpha<1\Big)$$

and

$$|T_{2,2}(g)| \leq \frac{1}{81} \qquad \Big(-\frac{1}{2}\leq\alpha<1\Big)$$

Proof: As $f\in\mathbb{M}(\alpha,\frac{1}{3},2)$ and by the definition, $T_{2,2}(h)=(a_2^2-a_3^2)$
By using the values of a_2,a_3 from (4.3), (4.4) that

$$\begin{aligned}T_{2,2}(h) =& \frac{1}{4}(1-\alpha)^2c_1^2-\frac{1}{36}[(1-\alpha)^2c_1^4+c_2^2+2(1-\alpha)c_1^2c_2]\\ =& \frac{1}{36}(1-\alpha)^2\Big[-(1-\alpha)^2c_1^4-2(1-\alpha)c_1^2c_2+9c_1^2-c_2^2\Big]\end{aligned}$$

By applying the triangle inequality and the using Lemmas 3.1, 3.2 and 3.3, we obtain

$$|a_2^2-a_3^2| \leq \frac{2}{9}(1-\alpha)^2\Big[2\alpha^2-6\alpha+9\Big] \qquad \Big(-\frac{1}{2}\leq\alpha<1\Big)$$

and also,

$$\begin{aligned}T_{2,2}(g) =& (d_2^2-d_3^2)\\ |T_{2,2}(g)| \leq& \Big(\frac{1}{9}\Big)^2 \quad (Using\ \ (4.11)\ \ and\ \ d_2=0)\\ \implies |T_{2,2}(g)| \leq& \frac{1}{81}.\end{aligned}$$

□

Theorem 4.3 *If* $f = h + \bar{g} \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$*, then*

$$|a_3^2 - a_2a_4| \leq \begin{cases} \dfrac{1}{18}(1-\alpha)^2\left(\alpha^2 - 2\alpha + 4\right), & -\dfrac{1}{2} \leq \alpha < \dfrac{1}{2}, \\ \dfrac{1}{18}(1-\alpha)^2\left(-\alpha^2 + \alpha + 3\right), & \dfrac{1}{2} \leq \alpha < 1. \end{cases}$$

and

$$|d_3^2 - d_2d_4| \leq \frac{1}{81}.$$

Proof: As $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$, it follows that $(a_3^2 - a_2a_4)$.
By using the values of a_2, a_3, a_4 from (4.3), (4.4), (4.5) that

$$\begin{aligned}(a_3^2 - a_2a_4) =& \frac{1}{36}(1-\alpha)^2\Big[(1-\alpha)^2c_1^4 + c_2^2 + 2(1-\alpha)c_1^2c_2\Big] - \frac{1}{48}(1-\alpha)^2\Big[(1-\alpha)^2c_1^4 + 3(1-\alpha)c_1^2c_2 \\ &+ 2c_1c_3\Big] \\ =& \frac{1}{144}(1-\alpha)^2\Big[(1-\alpha)^2c_1^4(1-\alpha)c_1^2c_2 + 4c_2^2 - 6c_1c_3\Big] \\ =& \frac{1}{144}(1-\alpha)^2\Big[c_1\Big((1-\alpha)^2c_1^3 - (1-\alpha)c_1c_2 - 2c_3\Big) + 4(c_2^2 - c_1c_3)\Big]\end{aligned}$$

From the triangle inequality and using Lemmas 3.1, 3.2, 3.3, we obtain
If $\Big(-\frac{1}{2} \leq \alpha < \frac{1}{2}\Big)$

$$\begin{aligned}|a_3^2 - a_2a_4| \leq& \frac{1}{144}(1-\alpha)^2\Big[2[4(\alpha^2 - 2\alpha + 2)] + 4(4)\Big] \\ \leq& \frac{1}{18}(1-\alpha)^2\Big[\alpha^2 - 2\alpha + 4\Big]\end{aligned}$$

If $\Big(\frac{1}{2} \leq \alpha < 1\Big)$

$$\begin{aligned}|a_3^2 - a_2a_4| \leq& \frac{1}{144}(1-\alpha)^2\Big[2[4(-\alpha^2 + \alpha + 1)] + 4(4)\Big] \\ \leq& \frac{1}{18}(1-\alpha)^2\Big[-\alpha^2 + \alpha + 3\Big]\end{aligned}$$

$$(d_3^2 - d_2d_4)$$

By using the values of $d_2 = 0, d_3, d_4$ from (4.11), (4.12) that

$$(d_3^2 - d_2d_4) = \frac{1}{81}$$

The triangle inequality implies that

$$|d_3^2 - d_2d_4| \leq \frac{1}{81}$$

□

Theorem 4.4 *Let* $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$*. Then*

$$|a_4^2 - a_3a_5| \leq \begin{cases} \dfrac{1}{180}(1-\alpha)^2\left(2\alpha^4 - 10\alpha^3 + 39\alpha^2 - 82\alpha + 57\right), & -\dfrac{1}{2} \leq \alpha < 0, \\ \dfrac{1}{180}(1-\alpha)^2\left(2\alpha^4 - 8\alpha^3 + 24\alpha^2 - 55\alpha + 43\right), & 0 \leq \alpha < 1. \end{cases}$$

and

$$|d_4^2 - d_3 d_5| \leq \begin{cases} \dfrac{1}{540}\left(7\alpha^2 - 18\alpha + 11\right), & -\dfrac{1}{2} \leq \alpha < -\dfrac{1}{7}, \\ \dfrac{1}{135}(1-\alpha), & -\dfrac{1}{7} \leq \alpha < 1. \end{cases}$$

Proof: As $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$, we have $(a_4^2 - a_3 a_5)$
By using the values of a_3, a_4, a_5 from (4.4), (4.5), (4.6) that

$$\begin{aligned} a_4^2 - a_3 a_5 =& \frac{1}{576}(1-\alpha)^2\Big[(1-\alpha)^4 c_1^6 + 6(1-\alpha)^3 c_4^4 c_2 + 9(1-\alpha)^2 c_1^2 c_2^2 + 4(1-\alpha)^2 c_1^3 c_3 + 12(1-\alpha) c_1 c_2 c_3 \\ & + 4c_3^2\Big] - \frac{1}{720}(1-\alpha)^2\Big[(1-\alpha)^4 c_1^6 + 7(1-\alpha)^3 c_1^4 c_2 + 8(1-\alpha)^2 c_1^3 c_3 + 9(1-\alpha)^2 c_1^2 c_2^2 \\ & + 6(1-\alpha) c_1^2 c_4 + 8(1-\alpha) c_1 c_2 c_3 + 3(1-\alpha) c_2^3 + 6c_2 c_4\Big] \\ =& \frac{1}{2880}(1-\alpha)^2\Big[(1-\alpha)^4 c_1^6 + 2(1-\alpha)^3 c_1^4 c_2 + 9(1-\alpha)^2 c_1^2 c_2^2 - 12(1-\alpha)^2 c_1^3 c_3 + 28(1-\alpha) c_1 c_2 c_3 \\ & - 24(1-\alpha) c_1^2 c_4 - 12(1-\alpha) c_2^3 + 20 c_3^2 - 24 c_2 c_4\Big] \\ =& \frac{1}{2880}(1-\alpha)^2\Big[(1-\alpha)^6 c_1^6 + (1-\alpha)^3 c_1^4 c_2 - 12(1-\alpha)^2 c_1^3 c_3 + (1-\alpha)^3 c_1^4 c_2 + 9(1-\alpha)^2 c_1^2 c_2^2 \\ & + 16(1-\alpha) c_1 c_2 c_3 + 20 c_3^2 - 20 c_2 c_4 - 4 c_2 c_4 - 24(1-\alpha) c_1^2 c_4 - 12(1-\alpha) c_2^3 + 12(1-\alpha) c_1 c_2 c_3\Big] \end{aligned}$$

By applying the triangle inequality and using the Lemmas 3.1, 3.2 and 3.3, we obtain

$$\begin{aligned} |a_4^2 - a_3 a_5| \leq & \frac{1}{2880}(1-\alpha)^2\Big[32(1-\alpha)^2[\alpha^2 - 2\alpha + 7] + 16(1-\alpha)[2\alpha^2 - 13\alpha + 19] + 96 + 192(1-\alpha) \\ & + 96(1-\alpha)\Big] \\ \leq & \frac{1}{180}(1-\alpha)^2\Big[2\alpha^4 - 10\alpha^3 + 39\alpha^2 - 82\alpha + 57\Big] \quad for\Big(-\frac{1}{2} \leq \alpha < 0\Big) \\ \leq & \frac{1}{2880}(1-\alpha)^2\Big[32(1-\alpha)^2[\alpha^2 - 2\alpha + 7] + 112(1-\alpha) + 96 + 192(1-\alpha) + 48(1-\alpha) \\ & + 16(1-\alpha)\Big] \\ \leq & \frac{1}{180}(1-\alpha)^2\Big[2\alpha^4 - 8\alpha^3 + 24\alpha^2 - 55\alpha + 43\Big] \quad for\Big(0 \leq \alpha < 1\Big) \end{aligned}$$

Further, by using the values of d_3, d_4, d_5 from (4.11), (4.12), (4.13) that

$$\begin{aligned} d_4^2 - d_3 d_5 =& \frac{1}{144}(1-\alpha)^2 c_1^2 - \frac{1}{270}(1-\alpha)\Big[(1-\alpha)c_1^2 + c_2\Big] \\ =& \frac{7}{2160}(1-\alpha)^2 c_1^2 - \frac{1}{270}(1-\alpha) c_2 \\ =& \frac{1}{2160}\Big[7(1-\alpha)^2 c_1^2 - 8(1-\alpha) c_2\Big] \\ =& \frac{1}{2160}\Big[-8(1-\alpha)[c_2 - \frac{7}{8}(1-\alpha)c_1^2]\Big] \end{aligned}$$

By applying the triangle inequality and using Lemmas 3.1, 3.2 and 3.3, we obtain

$$|d_4^2 - d_3 d_5| \leq \begin{cases} \dfrac{1}{540}\left(7\alpha^2 - 18\alpha + 11\right), & -\dfrac{1}{2} \leq \alpha < -\dfrac{1}{7}, \\ \dfrac{1}{135}(1-\alpha), & -\dfrac{1}{7} \leq \alpha < 1. \end{cases}$$

□

Theorem 4.5 *Let* $f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$. *Then*

$$|T_{4,2}(h)| \leq \begin{cases} \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 40\alpha^7 + 2240\alpha^6 - 20532\alpha^5 + 99917\alpha^4 - 293216\alpha^3 \\ \qquad +534846\alpha^2 - 572004\alpha + 287925\Big] \quad (-\frac{1}{2} \leq \alpha < 0) \\ \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 32\alpha^7 + 2144\alpha^6 - 20028\alpha^5 + 98156\alpha^4 - 289008\alpha^3 \\ \qquad +528765\alpha^2 - 567386\alpha + 286525\Big] \quad (0 \leq \alpha < \frac{1}{2}) \\ \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 32\alpha^7 + 1504\alpha^6 - 13884\alpha^5 + 77356\alpha^4 - 242208\alpha^3 \\ \qquad +474125\alpha^2 - 53226\alpha + 280885\Big] \quad (\frac{1}{2} \leq \alpha < 1) \end{cases}$$

and

$$|T_{4,2}(g)| \leq \begin{cases} \dfrac{1}{874800}\left[147\alpha^4 - 756\alpha^3 + 2354\alpha^2 - 3188\alpha + 12243\right], & -\dfrac{1}{2} \leq \alpha < -\dfrac{1}{7}, \\ \dfrac{1}{109350}\left[121\alpha^2 - 262\alpha + 1491\right], & -\dfrac{1}{7} \leq \alpha < 1. \end{cases}$$

Proof:

$$T_{4,2}(h) = (a_2^2 - a_3^2)^2 - (a_3a_4 - a_2a_5)^2 + (a_4^2 - a_3a_5)^2 - (a_2a_3 - a_3a_4)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5)$$

By using the values of a_2, a_3, a_4, a_5 from (4.3), (4.4), (4.5) and (4.6) that

$$\begin{aligned} (a_2^2 - a_3^2) =& \frac{1}{36}(1-\alpha)^2\Big[-(1-\alpha)^2c_1^4 - 2(1-\alpha)c_1^2c_2 + 9c_1^2 - c_2^2\Big] \\ (a_3a_4 - a_2a_5) =& \frac{1}{360}(1-\alpha)^2\Big[(1-\alpha)^3c_1^5 + (1-\alpha)^2c_1^3c_2 - 7(1-\alpha)c_1^2c_3 + 3(1-\alpha)c_1c_2^2 + 5c_2c_3 - 9c_1c_4\Big] \\ (a_4^2 - a_3a_5) =& \frac{1}{2880}(1-\alpha)^2\Big[(1-\alpha)^4c_1^6 + 2(1-\alpha)^3c_1^4c_2 + 9(1-\alpha)^2c_1^2c_2^2 - 12(1-\alpha)^2c_1^3c_3 \\ &+ 28(1-\alpha)c_1c_2c_3 - 24(1-\alpha)c_1^2c_4 - 12(1-\alpha)c_2^3 + 20c_3^2 - 24c_2c_4\Big] \\ (a_2a_3 - a_3a_4) =& \frac{1}{144}(1-\alpha)^2\Big[-(1-\alpha)^3c_1^5 - 4(1-\alpha)^2c_1^3c_2 - 2(1-\alpha)c_1^2c_3 - 3(1-\alpha)c_1c_2^2 + 12(1-\alpha)c_1^3 \\ &+ 12c_1c_2 - 2c_2c_3\Big] \\ (a_3^2 - a_2a_4) =& \frac{1}{144}(1-\alpha)^2\Big[(1-\alpha)^2c_1^4 - (1-\alpha)c_1^2c_2 + 4c_2^2 - 6c_1c_3\Big] \\ (a_2a_4 - a_3a_5) =& \frac{1}{720}(1-\alpha)^2\Big[-(1-\alpha)^4c_1^6 - 7(1-\alpha)^3c_1^4c_2 - 9(1-\alpha)^2c_1^2c_2^2 - 8(1-\alpha)^2c_1^3c_3 \\ &+ 15(1-\alpha)^2c_1^4 + 45(1-\alpha)c_1^2c_2 - 6(1-\alpha)c_1^2c_4 - 8(1-\alpha)c_1c_2c_3 - 3(1-\alpha)c_2^3 + 30c_1c_3 \\ &- 6c_2c_4\Big] \end{aligned}$$

By applying the triangle inequality and using lemmas 3.1, 3.2 and 3.3 this yields the bound for $(a_2^2 -$

$a_3^2), (a_3a_4 - a_2a_5), (a_4^2 - a_3a_5), (a_2a_3 - a_3a_4), (a_3^2 - a_2a_4)$ and $(a_2a_4 - a_3a_5)$.

$$\begin{cases} |a_2^2 - a_3^2| \leq \frac{2}{9}(1-\alpha)^2\Big[2\alpha^2 - 6\alpha + 9\Big] \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \\ |a_3a_4 - a_2a_5| \leq \frac{1}{90}(1-\alpha)^2\Big[-4\alpha^3 + 12\alpha 2 - 26\alpha + 2\Big] \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \\ |a_4^2 - a_3a_5| \leq \begin{cases} \dfrac{1}{180}(1-\alpha)^2\left(2\alpha^4 - 10\alpha^3 + 39\alpha^2 - 82\alpha + 57\right), -\dfrac{1}{2} \leq \alpha < 0, \\ \dfrac{1}{180}(1-\alpha)^2\left(2\alpha^4 - 8\alpha^3 + 24\alpha^2 - 55\alpha + 43\right), 0 \leq \alpha < 1, \end{cases} \\ |a_2a_3 - a_3a_4| \leq \frac{1}{18}\Big[-4\alpha^3 + 20\alpha^2 - 45\alpha + 36\Big] \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \\ |a_3^2 - a_2a_4| \begin{cases} \leq \dfrac{1}{18}(1-\alpha)^2\left(\alpha^2 - 2\alpha + 4\right), -\dfrac{1}{2} \leq \alpha < \dfrac{1}{2}, \\ \leq \dfrac{1}{18}(1-\alpha)^2\left(-\alpha^2 + \alpha + 3\right), \dfrac{1}{2} \leq \alpha < 1, \end{cases} \\ |a_2a_4 - a_3a_5| \leq \frac{1}{90}\Big[8\alpha^4 - 60\alpha^3 + 166\alpha^2 - 306\alpha + 141\Big] \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \end{cases} \tag{4.23}$$

If $\Big(-\frac{1}{2} \leq \alpha < 0\Big)$

$$\big|T_{4,2}(h)\big| \leq |a_2^2 - a_3^2|^2 + |a_3a_4 - a_2a_5|^2 + |a_4^2 - a_3a_5|^2 + |a_2a_3 - a_3a_4|^2 + 2|a_3^2 - a_2a_4||a_2a_4 - a_3a_5| \tag{4.24}$$

By using the bounds of $\big|(a_2^2 - a_3^2)\big|, \big|(a_3a_4 - a_2a_5)\big|, \big|(a_4^2 - a_3a_5)\big|, \big|(a_2a_3 - a_3a_4)\big|, \big|(a_3^2 - a_2a_4)\big|, \big|(a_2a_4 - a_3a_5)\big|$ from (4.23) that

$$\begin{aligned} \Big|T_4(2)(h)\Big| \leq & \left|\frac{2}{9}(1-\alpha)^2\Big[2\alpha^2 - 6\alpha + 9\Big]\right|^2 + \left|\frac{1}{90}(1-\alpha)^2\Big[-4\alpha^3 + 12\alpha 2 - 26\alpha + 2\Big]\right|^2 + \left|\frac{1}{180}(1-\alpha)^2\Big[2\alpha^4\right. \\ & \left.- 10\alpha^3 + 39\alpha^2 - 82\alpha + 57\Big]\right|^2 + \left|\frac{1}{18}\Big[-4\alpha^3 + 20\alpha^2 - 45\alpha + 36\Big]\right|^2 + 2\left|\frac{1}{18}(1-\alpha)^2\Big[\alpha^2 - 2\alpha\right. \\ & \left.+ 4\Big]\right|\left|\frac{1}{90}\Big[8\alpha^4 - 60\alpha^3 + 166\alpha^2 - 306\alpha + 141\Big]\right| \\ \leq & \frac{4}{81}(1-\alpha)^4\Big[4\alpha^4 - 24\alpha^3 + 72\alpha^2 - 108\alpha + 81\Big] + \frac{1}{8100}(1-\alpha)^4\Big[16\alpha^6 - 96\alpha^5 + 352\alpha^4 - 840\alpha^3 \\ & + 1324\alpha^2 - 1404\alpha + 729\Big] + \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 40\alpha^7 + 256\alpha^6 - 1108\alpha^5 + 3389\alpha^4 \\ & - 7536\alpha^3 + 11170\alpha^2 - 9348\alpha + 3249\Big] + \frac{1}{324}(1-\alpha)^4\Big[16\alpha^6 - 160\alpha^5 + 760\alpha^4 - 2088\alpha^3 \\ & + 3465\alpha^2 - 3240\alpha + 1296\Big] + \frac{1}{810}(1-\alpha)^4\Big[8\alpha^6 - 76\alpha^5 + 318\alpha^4 - 878\alpha^3 + 1417\alpha^2 - 1506\alpha \\ & + 564\Big] \\ \leq & \frac{1}{32400}(1-\alpha)^4\Big[1600(\alpha^4 - 24\alpha^3 + 72\alpha^2 - 108\alpha + 81)\Big] + \Big[4\Big(16\alpha^6 - 96\alpha^5 + 352\alpha^4 - 840\alpha^3 \\ & + 1324\alpha^2 - 1404\alpha + 729\Big)\Big] + \Big[4\alpha^8 - 40\alpha^7 + 256\alpha^6 - 1108\alpha^5 + 3389\alpha^4 - 7536\alpha^3 + 11170\alpha^2 \\ & - 9348\alpha + 3249\Big] + \Big[100(16\alpha^6 - 160\alpha^5 + 760\alpha^4 - 2088\alpha^3 + 3465\alpha^2 - 3240\alpha + 1296)\Big] \\ & + \Big[40(8\alpha^6 - 76\alpha^5 + 318\alpha^4 - 878\alpha^3 + 1417\alpha^2 - 1506\alpha + 564)\Big] \end{aligned}$$

$$\begin{aligned} \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 40\alpha^7 + 2240\alpha^6 - 20532\alpha^5 + 99917\alpha^4 - 293216\alpha^3 + 534846\alpha^2 - 572004\alpha \\ & + 287925\Big] \end{aligned}$$

$$\begin{aligned}\big|T_4(2)[h]\big| \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8-40\alpha^7+2240\alpha^6-20532\alpha^5+99917\alpha^4-293216\alpha^3+534846\alpha^2\\ & -572004\alpha+287925\Big]\ \ \Big(-\frac{1}{2}\leq\alpha<0\Big)\end{aligned} \tag{4.25}$$

If $\left(0\leq\alpha<\frac{1}{2}\right)$
By using the bounds from (4.23) in equation (4.24)

$$\begin{aligned}\big|T_{4,2}(h)\big| \leq & \left|\frac{2}{9}(1-\alpha)^2\Big[2\alpha^2-6\alpha+9\Big]\right|^2+\left|\frac{1}{90}(1-\alpha)^2\Big[-4\alpha^3+12\alpha2-26\alpha+2\Big]\right|^2+\left|\frac{1}{180}(1-\alpha)^2\Big[2\alpha^4\right.\\ & \left.-8\alpha^3+24\alpha^2-55\alpha+43\Big]\right|^2+\left|\frac{1}{18}\Big[-4\alpha^3+20\alpha^2-45\alpha+36\Big]\right|^2+2\left|\frac{1}{18}(1-\alpha)^2\Big[\alpha^2-2\alpha\right.\\ & \left.+4\Big]\right|\left|\frac{1}{90}\Big[8\alpha^4-60\alpha^3+166\alpha^2-306\alpha+141\Big]\right|\\ \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8-32\alpha^7+2144\alpha^6-20028\alpha^5+98156\alpha^4-289008\alpha^3+528765\alpha^2\\ & -567386\alpha+286525\Big]\end{aligned}$$

$$\begin{aligned}\big|T_{4,2}(h)\big| \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8-32\alpha^7+2144\alpha^6-20028\alpha^5+98156\alpha^4-289008\alpha^3+528765\alpha^2\\ & -567386\alpha+286525\Big]\quad\Big(0\leq\alpha<\frac{1}{2}\Big)\end{aligned} \tag{4.26}$$

If $\left(\frac{1}{2}\leq\alpha<1\right)$
By using the bounds from (4.23) in equation in (4.24)

$$\begin{aligned}\big|T_{4,2}(h)\big| \leq & \left|\frac{2}{9}(1-\alpha)^2\Big[2\alpha^2-6\alpha+9\Big]\right|^2+\left|\frac{1}{90}(1-\alpha)^2\Big[-4\alpha^3+12\alpha2-26\alpha+2\Big]\right|^2+\left|\frac{1}{180}(1-\alpha)^2\Big[2\alpha^4\right.\\ & \left.-8\alpha^3+24\alpha^2-55\alpha+43\Big]\right|^2+\left|\frac{1}{18}\Big[-4\alpha^3+20\alpha^2-45\alpha+36\Big]\right|^2+2\left|\frac{1}{18}(1-\alpha)^2\Big[-\alpha^2+\alpha\right.\\ & \left.+3\Big]\right|\left|\frac{1}{90}\Big[8\alpha^4-60\alpha^3+166\alpha^2-306\alpha+141\Big]\right|\\ \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8-32\alpha^7+1504\alpha^6-13884\alpha^5+77356\alpha^4-242208\alpha^3+474125\alpha^2-53226\alpha\\ & +280885\Big]\end{aligned}$$

$$\begin{aligned}\big|T_{4,2}(h)\big| \leq & \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8-32\alpha^7+1504\alpha^6-13884\alpha^5+77356\alpha^4-242208\alpha^3+474125\alpha^2-53226\alpha\\ & +280885\Big]\ \Big(\frac{1}{2}\leq\alpha<1\Big)\end{aligned} \tag{4.27}$$

By equations (4.25), (4.26) and (4.27) that

$$|T_{4,2}(h)| \leq \begin{cases} \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 40\alpha^7 + 2240\alpha^6 - 20532\alpha^5 + 99917\alpha^4 - 293216\alpha^3 \\ \qquad +534846\alpha^2 - 572004\alpha + 287925\Big] \quad (-\frac{1}{2} \leq \alpha < 0) \\ \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 32\alpha^7 + 2144\alpha^6 - 20028\alpha^5 + 98156\alpha^4 - 289008\alpha^3 \\ \qquad +528765\alpha^2 - 567386\alpha + 286525\Big] \quad (0 \leq \alpha < \frac{1}{2}) \\ \frac{1}{32400}(1-\alpha)^4\Big[4\alpha^8 - 32\alpha^7 + 1504\alpha^6 - 13884\alpha^5 + 77356\alpha^4 - 242208\alpha^3 \\ \qquad +474125\alpha^2 - 53226\alpha + 280885\Big] \quad (\frac{1}{2} \leq \alpha < 1) \end{cases} \tag{4.28}$$

$$\begin{aligned} T_{4,2}(g) =& (d_2^2 - d_3^2)^2 - (d_3d_4 - d_2d_5)^2 + (d_4^2 - d_3d_5)^2 - (d_2d_3 - d_3d_4)^2 + 2(d_3^2 - d_2d_4)(d_2d_4 - d_3d_5) \\ =& d_3^2\big[1 - 2d_4^2 - 2d_3d_5\big] + \big[d_4^2 - d_3b_5\big]^2 \end{aligned}$$

By using the values of d_3, d_4, d_5 from (4.11), (4.12) and (4.13) that

$$\begin{aligned} (d_3) =& \frac{1}{9} \\ (1 - 2d_4^2 - 2d_3d_5) =& \frac{1}{1080}\Big[-23(1-\alpha)^2c_1^2 - 8(1-\alpha)c_2 + 1080\Big] \\ (d_4^2 - d_3d_5) =& \frac{1}{2160}\Big[7(1-\alpha)^2c_1^2 - 8(1-\alpha)c_2\Big] \end{aligned}$$

By applying the triangle inequality and using the Lemmas 3.1 and 3.2 hence, we have the bound for the $(d_3), (1 - 2d_4^2 - 2d_3d_5), (d_4^2 - d_3d_5)$.

$$\begin{cases} |d_3| \leq \frac{1}{9} \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \\ |1 - 2d_4^2 - 2d_3d_5| \leq \frac{1}{270}\Big[23\alpha^2 - 50\alpha + 297\Big] \quad \Big(-\frac{1}{2} \leq \alpha < 1\Big) \\ |d_4^2 - d_3d_5| \leq \begin{cases} \frac{1}{540}\Big[7\alpha^2 - 18\alpha + 11\Big] \quad \Big(-\frac{1}{2} \leq \alpha < -\frac{1}{7}\Big) \\ \frac{1}{135}(1-\alpha) \quad \Big(-\frac{1}{7} \leq \alpha < 1\Big) \end{cases} \end{cases} \tag{4.29}$$

If $\Big(-\frac{1}{2} \leq \alpha < -\frac{1}{7}\Big)$

$$\big|T_{4,2}(g)\big| \leq \big|d_3\big|^2\big|\big[1 - 2d_4^2 - 2d_3d_5\big]\big| + \big|\big[d_4^2 - d_3d_5\big]\big|^2 \tag{4.30}$$

By using the bounds of $|d_3|, |1 - 2d_4^2 - 2d_3d_5|, |d_4^2 - d_3d_5|$ from (4.29) that

$$\begin{aligned} \Big|T_{4,2}(g)\Big| \leq& \left[\frac{1}{9}\right]^2\left[\frac{1}{270}\Big[23\alpha^2 - 50\alpha + 297\Big]\right] + \left[\frac{1}{540}\Big[7\alpha^2 - 18\alpha + 11\Big]\right]^2 \\ \leq& \frac{1}{81}\cdot\frac{1}{270}\Big[23\alpha^2 - 50\alpha + 297\Big] + \frac{1}{291600}\Big[49\alpha^4 - 252\alpha^3 + 478\alpha^2 - 396\alpha + 121\Big] \\ \leq& \frac{1}{874800}\Big[147\alpha^4 - 756\alpha^3 + 2354\alpha^2 - 3188\alpha + 12243\Big] \end{aligned}$$

$$\Big|T_{4,2}(g)\Big| \leq \frac{1}{874800}\Big[147\alpha^4 - 756\alpha^3 + 2354\alpha^2 - 3188\alpha + 12243\Big] \quad \Big(-\frac{1}{2} \leq \alpha < -\frac{1}{7}\Big) \tag{4.31}$$

If $\Big(-\frac{1}{7} \leq \alpha < 1\Big)$
By using the bounds of $|d_3|, |1-2d_4^2-2d_3d_5|, |d_4^2-d_3d_5|$ from (4.29) that

$$\begin{aligned}\Big|T_{4,2}(g)\Big| \leq & \left[\frac{1}{9}\right]^2\left[\frac{1}{270}\Big[23\alpha^2-50\alpha+297\Big]\right]+\left[\frac{1}{135}(1-\alpha)\right]^2\\ \leq & \frac{1}{81}.\frac{1}{270}\Big[23\alpha^2-50\alpha+297\Big]+\frac{1}{18225}(1-\alpha)^2\\ \leq & \frac{1}{109350}\Big[121\alpha^2-262\alpha+1491\Big]\end{aligned}$$

$$\Big|T_{4,2}(g)\Big| \leq \frac{1}{109350}\Big[121\alpha^2-262\alpha+1491\Big] \qquad \Big(-\frac{1}{7} \leq \alpha < 1\Big) \tag{4.32}$$

From equation (4.31) and (4.32), we get

$$|T_{4,2}(g)| \leq \begin{cases} \dfrac{1}{874800}\left[147\alpha^4-756\alpha^3+2354\alpha^2-3188\alpha+12243\right], & -\dfrac{1}{2} \leq \alpha < -\dfrac{1}{7},\\ \dfrac{1}{109350}\left[121\alpha^2-262\alpha+1491\right], & -\dfrac{1}{7} \leq \alpha < 1.\end{cases}$$

□

By setting α = {-1/2, -1/4, 0, 1/4, 1/2} in 4.1, we obtain

Table 1: Upper bounds of fourth order determinants $T_{4,1}(f)$.

$T_{4,1}(f)$	$\alpha=-\frac{1}{2}$	$\alpha=-\frac{1}{4}$	$\alpha=0$	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{2}$
$\lvert T_{4,1}(h)\rvert$	≤ 41.732042	≤ 21.991658	≤ 8.115741	≤ 3.981133	≤ 2.026596
$\lvert T_{4,1}(g)\rvert$	≤ 20.364390	≤ 12.731099	≤ 7.876736	≤ 4.301758	≤ 2.240170

Let $f = h + \bar{g} \in f \in \mathbb{M}(\alpha, \frac{1}{3}, 2)$ be of the form (1.1).

Table 2: Upper bounds of fourth order determinants $T_{4,2}(f)$.

$T_{4,2}(f)$	$\alpha=-\frac{1}{2}$	$\alpha=-\frac{1}{4}$	$\alpha=0$	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{2}$
$\lvert T_{4,2}(h)\rvert$	≤ 117.376917	≤ 35.366202	≤ 8.843364	≤ 1.695068	≤ 0.669274
$\lvert T_{4,2}(g)\rvert$	≤ 0.016609	≤ 0.015089	≤ 0.013635	≤ 0.013105	≤ 0.012714

Concluding Remark: From the above tables we conclude that the bounds of functionals $|T_{4,1}(h)|$, $|T_{4,1}(g)|$, $|T_{4,2}(h)|$ and $|T_{4,2}(g)|$ decreasing when α increasing from -1/2 to 1/2.

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