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Coupled Fixed Point Theorem in $\mathrm{G_{JS}}$-Metric Space

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ABSTRACT: This article aims at constructing a coupled fixed point theorem in a new metric space known as $\mathrm{G_{JS}}$- metric space through the process of generalization. The process is initialized by proving a lemma and the main result establishes the existence and uniqueness of coupled fixed point which in turn is supported by an example.

Keywords: $\mathrm{G_{JS}}$-metric space, complete $\mathrm{G_{JS}}$-metric space, coupled fixed point.

Contents

1. Introduction

Out of wide range of applications of fixed point theory in non-linear analysis [2, 9, 3, 4], coupled fixed point theorems are one of the most interesting concepts introduced by Guo and Lakshmikantham [5]. Coupled fixed point theorems can be generalized through various metric spaces [8, 6, 10, 1, 12, 11]. The main aim of this article is to generalize a coupled fixed point theorem through $\mathrm{G_{JS}}$-metric space [12]. In this article, we establish a theorem in which a map satisfying certain conditions on a complete $\mathrm{G_{JS}}$-metric space has a coupled fixed point in it. This is the generalization of coupled fixed point result given by Kushal Roy, Mantu Saha and Ismat Beg [7] through $\mathrm{G_{JS}}$ -metric space. We also establish here that there exists only one such fixed point satisfying the given conditions. A suitable example is provided to support the theorem.

2. Preliminaries

Definition 2.1 *Assume that E is a non-empty set and $\mathrm{G_{JS}}$: $E^3 \to [0, \infty]$ is a mapping fulfilling the following conditions:*
(G_{JS}1) $\mathrm{G_{JS}}(\chi, \psi, \xi) = 0$ *if and only if* $\chi = \psi = \xi$.
(G_{JS}2) $0 < \mathrm{G_{JS}}(\chi, \chi, \psi)$ *for all* $\chi, \psi \in \mathrm{E}$ *with* $\chi \neq \psi$.
(G_{JS}3) $\mathrm{G_{JS}}(\chi, \chi, \psi) < \mathrm{G_{JS}}(\chi, \psi, \xi)$ *for all* $\chi, \psi, \xi \in \mathrm{E}$ *with* $\psi \neq \xi$.
(G_{JS}4) $\mathrm{G_{JS}}(\chi, \psi, \xi) = \mathrm{G_{JS}}(\sigma(\chi, \psi, \xi))$ *for all* $\chi, \psi, \xi \in \mathrm{E}$ *where* $\sigma(\chi,\psi,\xi)$ *is a permutation of the set* $\{\chi,\psi,\xi\}$ *and*
(G_{JS}5) there is a constant $c > 0$ such that for $(\chi,\psi,\xi) \in E^3$ and $\langle\chi_n\rangle \in \mathcal{G}$ ($\mathrm{G_{JS}}$, E, χ),

$$\mathrm{G_{JS}}(\chi, \psi, \xi) \leq \mathrm{c} \limsup_{n\to\infty} \mathrm{G_{JS}}(\chi_n, \psi, \xi)$$

where $\mathcal{G}(\mathrm{G_{JS}}, E, \chi) = \left\{\langle\chi_n\rangle \subset \mathrm{E} : \lim_{n\to\infty} \mathrm{G_{JS}}(\chi_n, \chi, \chi) = 0\right\}$
Then, the mapping $\mathrm{G_{JS}}$ is called a $\mathrm{G_{JS}}$- metric on E and the pair $(E, \mathrm{G_{JS}})$ is called a G_{JS}-metric space.

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Definition 2.2 *Let (E,* $\mathrm{G_{JS}}$*) be a* $\mathrm{G_{JS}}$*-metric space. A sequence* $\langle \chi_{\mathrm{n}} \rangle$ *in E is said to be* ***G_{JS}-convergent*** *to* $\chi \in E$ *if* $\lim\limits_{\mathrm{n,\, m} \to \infty} \mathrm{G_{JS}}(\chi, \chi_{\mathrm{n}}, \chi_{\mathrm{m}}) = 0$.
That is, for any $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $\mathrm{G_{JS}}(\chi, \chi_{\mathrm{n}}, \chi_{\mathrm{m}}) < \varepsilon$ *for all* $n, m \geq N$**.**
Here, 'χ*' is called the limit of the sequence* $\langle \chi_{\mathrm{n}} \rangle$ *and we write* $\lim\limits_{\mathrm{n} \to \infty} \chi_{\mathrm{n}} = \chi$ *or* $\chi_{\mathrm{n}} \to \chi$ *as* $\mathrm{n} \to \infty$*.*

Definition 2.3 *Let (E,* $\mathrm{G_{JS}}$*) be a* $\mathrm{G_{JS}}$*- metric space. A sequence* $\langle \chi_{\mathrm{n}} \rangle$ *is called* ***G_{JS}-Cauchy*** *if for any* $\varepsilon > 0$ *and for all* $r, s, t \geq N$*, there exists* $N \in \mathbb{N}$ *such that* $\mathrm{G_{JS}}(\chi_{\mathrm{r}}, \chi_{\mathrm{s}}, \chi_{\mathrm{t}}) < \varepsilon$*, that is* $\mathrm{G_{JS}}(\chi_{\mathrm{r}}, \chi_{\mathrm{s}}, \chi_{\mathrm{t}}) \to 0$ *as* $r, s, t \to \infty$*.*

Definition 2.4 *A* $\mathrm{G_{JS}}$*- metric space is said to be* $\mathrm{G_{JS}}$***- complete*** *if every* $\mathrm{G_{JS}}$*- Cauchy sequence in E is* $\mathrm{G_{JS}}$*-convergent in E.*

Definition 2.5 *Let E be a non-empty set and* $\mu : \mathrm{E}^2 \to \mathrm{E}$ *be a map. An element* $(\chi, \psi) \in \mathrm{E}^2$ *is called a* ***coupled fixed point*** *of* μ *if* $\mu(\chi, \psi) = \chi$ *and* $\mu(\psi, \chi) = \psi$*.*

Lemma 2.1 Let $\mathrm{p} = \sup\{\mathrm{p}_\alpha \in \mathrm{R} : \alpha \geq 1\} < \infty$*. For any* $M > 0$*, if* $\mathrm{p}_\alpha \leq \mathrm{M}$ for $\alpha \geq 1$ *then* $p \leq M$*.*

Proof: For $\mathrm{n} \in \mathrm{N}$, we can find $\mathrm{t} \geq 1$ such that $\mathrm{p} < \mathrm{p_t} + \frac{1}{\mathrm{n}} \leq \mathrm{M} + \frac{1}{\mathrm{n}}$.
As $\mathrm{n} \to \infty$, we get $\mathrm{p} \leq \mathrm{M}$. □

Note 1 *For* $(\chi_0, \psi_0) \in \mathrm{E}^2$ *and* $\mu : \mathrm{E}^2 \to \mathrm{E}$*, we define*

$$\Delta(\mathrm{G_{JS}}, \mu, (\chi_0, \psi_0)) = \left\{\mathrm{G_{JS}}\left(\mu^{\mathrm{i}}(\chi_0, \psi_0), \mu^{\mathrm{i}}(\chi_0, \psi_0), \mu^{\mathrm{j}}(\chi_0, \psi_0) : \mathrm{i, j} \in \mathrm{N}\right)\right\} \text{ and}$$

$$\Delta(\mathrm{G_{JS}}, \mu, (\psi_0, \chi_0)) = \left\{\mathrm{G_{JS}}\left(\mu^{\mathrm{i}}(\psi_0, \chi_0), \mu^{\mathrm{i}}(\psi_0, \chi_0), \mu^{\mathrm{j}}(\psi_0, \chi_0) : \mathrm{i, j} \in \mathrm{N}\right)\right\}$$

3. Main Results

Theorem 3.1 *Assume that* (E, $\mathrm{G_{JS}}$) *is a* $\mathrm{G_{JS}}$*- metric space which is* $\mathrm{G_{JS}}$*- complete and a map* $\mu : \mathrm{E}^2 \to \mathrm{E}$ *fulfils the condition,*

$$\mathrm{G_{JS}}(\mu(\chi, \psi), \mu(\chi, \psi), \mu(\alpha, \beta)) \leq \lambda \max\{\mathrm{G_{JS}}(\chi, \chi, \alpha), \mathrm{G_{JS}}(\psi, \psi, \beta)\} \tag{3.1}$$

for every $\chi, \psi, \alpha, \beta \in \mathrm{E}$*,* $\lambda \in [0, 1)$*.*
If we can find (χ_0, ψ_0) *in* E^2 *such that* $\Delta(\mathrm{G_{JS}}, \mu, (\chi_0, \psi_0)) < \infty$ *and* $\Delta(\mathrm{G_{JS}}, \mu, (\psi_0, \chi_0)) < \infty$*, then* μ *has* coupled fixed point *in* E.

Proof:
We denote,

$$\mu^2(\chi_0, \psi_0) = \mu(\mu(\chi_0, \psi_0), \mu(\psi_0, \chi_0)),$$
$$\mu^2(\psi_0, \chi_0) = \mu(\mu(\psi_0, \chi_0), \mu(\chi_0, \psi_0))$$

and

$$\mu^3(\chi_0, \psi_0) = \mu\left(\mu^2(\chi_0, \psi_0), \mu^2(\psi_0, \chi_0)\right),$$
$$\mu^3(\psi_0, \chi_0) = \mu\left(\mu^2(\psi_0, \chi_0), \mu^2(\chi_0, \psi_0)\right).$$

Proceeding in a similar manner, we get

$$\mu^{\mathrm{n}+1}(\chi_0, \psi_0) = \mu(\mu^{\mathrm{n}}(\chi_0, \psi_0), \mu^{\mathrm{n}}(\psi_0, \chi_0)), \tag{3.2}$$

$$\mu^{\mathrm{n}+1}(\psi_0, \chi_0) = \mu(\mu^{\mathrm{n}}(\psi_0, \chi_0), \mu^{\mathrm{n}}(\chi_0, \psi_0)). \tag{3.3}$$

Now for $\mathrm{i, j} \geq 1$, $\mathrm{n} \in \mathrm{N}$ and using (3.1),

$$\mathrm{G_{JS}}\left(\mu^{\mathrm{n}+\mathrm{i}}(\chi_0, \psi_0), \mu^{\mathrm{n}+\mathrm{i}}(\chi_0, \psi_0), \mu^{\mathrm{n}+\mathrm{j}}(\chi_0, \psi_0)\right)$$

$$= \mathrm{G_{JS}}\begin{pmatrix} \mu\left(\mu^{n-1+i}(\chi_0,\psi_0),\ \mu^{n-1+i}(\psi_0,\chi_0)\right), \\ \mu\left(\mu^{n-1+i}(\chi_0,\psi_0),\ \mu^{n-1+i}(\psi_0,\chi_0)\right), \\ \mu\left(\mu^{n-1+j}(\chi_0,\psi_0),\ \mu^{n-1+j}(\psi_0,\chi_0)\right) \end{pmatrix}$$

$$\leq \lambda \max\left\{ \begin{array}{l} \mathrm{G_{JS}}\left(\mu^{n-1+i}(\chi_0,\psi_0),\ \mu^{n-1+i}(\chi_0,\psi_0),\ \mu^{n-1+j}(\chi_0,\psi_0)\right), \\ \mathrm{G_{JS}}\left(\mu^{n-1+i}(\psi_0,\chi_0),\ \mu^{n-1+i}(\psi_0,\chi_0),\ \mu^{n-1+j}(\psi_0,\chi_0)\right) \end{array} \right\}.$$

For any $t \geq 0$and $i, j \geq 1$, let

$$\Delta\left(\mathrm{G_{JS}},\ \mu^{t+1},\ (\chi_0,\psi_0)\right) = \sup\left\{\mathrm{G_{JS}}\left(\mu^{t+i}(\chi_0,\psi_0),\ \mu^{t+i}(\chi_0,\psi_0), \mu^{t+j}(\chi_0,\psi_0)\right)\right\}$$

$$\text{and } \Delta\left(\mathrm{G_{JS}},\ \mu^{t+1},\ (\psi_0,\chi_0)\right) = \sup\left\{\mathrm{G_{JS}}\left(\mu^{t+i}(\psi_0,\chi_0),\ \mu^{t+i}(\psi_0,\chi_0), \mu^{t+j}(\psi_0,\chi_0)\right)\right\}.$$

Then, $\mathrm{G_{JS}}\left(\mu^{n+i}(\chi_0,\psi_0),\ \mu^{n+i}(\chi_0,\psi_0),\ \mu^{n+j}(\chi_0,\psi_0)\right) \leq \lambda \max\{\Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)),\ \Delta(\mathrm{G_{JS}},\mu^n,(\psi_0,\chi_0))\}$.
Since for any $t \geq 1$, $\Delta\left(\mathrm{G_{JS}},\ \mu^{t+1},\ (\chi_0,\psi_0)\right) \leq \Delta\left(\mathrm{G_{JS}},\ \mu,\ (\chi_0,\psi_0)\right) < \infty$ and from Lemma (2.1), we get

$$\Delta\left(\mathrm{G_{JS}},\ \mu^{n+1},\ (\chi_0,\psi_0)\right) \leq \lambda \max\{\Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)),\ \Delta(\mathrm{G_{JS}},\mu^n,(\psi_0,\chi_0))\} \tag{3.4}$$

and

$$\Delta\left(\mathrm{G_{JS}},\ \mu^{n+1},\ (\psi_0,\chi_0)\right) \leq \lambda \max\{\Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)),\ \Delta(\mathrm{G_{JS}},\mu^n,(\psi_0,\chi_0))\}. \tag{3.5}$$

Let $M_n = \max\{\Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)),\ \Delta(\mathrm{G_{JS}},\mu^n,(\psi_0,\chi_0))\}$.
Then (3.4) becomes,

$$M_{n+1} \leq \lambda M_n \leq \lambda^2 M_{n-1} \cdots \leq \lambda^n M_1.$$

As $M_1 < \infty$, we obtain $\lim\limits_{n\to\infty} M_{n+1} = 0 \Rightarrow \lim\limits_{n\to\infty} M_n = 0$, showing that

$$\lim_{n\to\infty} \Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)) = 0 = \Delta(\mathrm{G_{JS}},\mu^n,(\psi_0,\chi_0)).$$

Now for $1 \leq n < m$, as $n \to \infty$ we observe that
$\mathrm{G_{JS}}(\mu^n(\chi_0,\psi_0),\ \mu^n(\chi_0,\psi_0),\ \mu^m(\chi_0,\psi_0)) \leq \Delta(\mathrm{G_{JS}},\mu^n,(\chi_0,\psi_0)) \to 0$, proving that $\langle\mu^n(\chi_0,\psi_0)\rangle$ is $\mathrm{G_{JS}}$- Cauchy in E.
As E is $\mathrm{G_{JS}}$- complete, we can find $\xi_1, \xi_2 \in$ E in such a way that

$$\mu^n(\chi_0,\psi_0) \to \xi_1 \text{ and } \mu^n(\psi_0,\chi_0) \to \xi_2 \text{ as } n \to \infty. \tag{3.6}$$

Therefore, using equation (3.1), we observe that
$\mathrm{G_{JS}}(\mu(\xi_1,\xi_2),\ \mu(\xi_1,\xi_2),\ \mu^n(\chi_0,\psi_0)) = \mathrm{G_{JS}}\left(\mu(\xi_1,\xi_2),\ \mu(\xi_1,\xi_2), \mu\left(\mu^{n-1}(\chi_0,\psi_0),\ \mu^{n-1}(\psi_0,\chi_0)\right)\right)$

$$\leq \lambda \max\left\{\mathrm{G_{JS}}\left(\xi_1,\xi_1,\ \mu^{n-1}(\chi_0,\psi_0)\right),\ \mathrm{G_{JS}}\left(\xi_2,\xi_2,\ \mu^{n-1}(\psi_0,\chi_0)\right)\right\}.$$

Using (3.6), we get
$\lim\limits_{n\to\infty} \mathrm{G_{JS}}(\mu(\xi_1,\xi_2),\ \mu(\xi_1,\xi_2),\ \mu^n(\chi_0,\psi_0)) = 0$, showing that $\lim\limits_{n\to\infty} \mu^n(\chi_0,\psi_0) = \mu(\xi_1,\xi_2) = \xi_1$.
Therefore, $\mu(\xi_1,\xi_2) = \xi_1$.
In a similar manner, we can prove that $\mu(\xi_2,\xi_1) = \xi_2$.
Hence μ has (ξ_1,ξ_2)as its coupled fixed point in E.
To prove the uniqueness of coupled fixed point of μ, assume that (ξ_1,ξ_2) and (ξ_1^*,ξ_2^*) are coupled fixed points of μ such that $\mathrm{G_{JS}}(\xi_1,\xi_1,\xi_1^*) < \infty$ and $\mathrm{G_{JS}}(\xi_2,\xi_2,\xi_2^*) < \infty$.
Then, $\mu(\xi_1,\xi_2) = \xi_1$, $\mu(\xi_2,\xi_1) = \xi_2$ and $\mu(\xi_1^*,\xi_2^*) = \xi_1^*$, $\mu(\xi_2^*,\xi_1^*) = \xi_2^*$.
Now, $\mathrm{G_{JS}}(\xi_1,\xi_1,\xi_1^*) = \mathrm{G_{JS}}(\mu(\xi_1,\xi_2),\ \mu(\xi_1,\xi_2),\ \mu(\xi_1^*,\xi_2^*))$

$$\leq \lambda \max\{\mathrm{G_{JS}}(\xi_1,\xi_1,\xi_1^*),\ \mathrm{G_{JS}}(\xi_2,\xi_2,\xi_2^*)\}. \tag{3.7}$$

Also, $\mathrm{G_{JS}}(\xi_2,\xi_2,\xi_2^*) = \mathrm{G_{JS}}(\mu(\xi_2,\xi_1),\ \mu(\xi_2,\xi_1),\ \mu(\xi_2^*,\xi_1^*))$

$$\leq \lambda \max\{\mathrm{G_{JS}}(\xi_2,\xi_2,\xi_2^*),\ \mathrm{G_{JS}}(\xi_1,\xi_1,\xi_1^*)\}. \tag{3.8}$$

Let $\mathrm{M}^* = \max\{\mathrm{G_{JS}}(\xi_1, \xi_1, \xi_1^*), \mathrm{G_{JS}}(\xi_2, \xi_2, \xi_2^*)\}$.
Then from (3.7) and (3.8) we get, $\mathrm{M}^* \le \lambda \mathrm{M}^*$.
Since $\lambda \in [0, 1)$, the above inequality is possible only when $\mathrm{M}^* = 0$.
This implies $\mathrm{G_{JS}}(\xi_1, \xi_1, \xi_1^*) = 0$ and $\mathrm{G_{JS}}(\xi_2, \xi_2, \xi_2^*) = 0$.
Therefore $\xi_1 = \xi_1^*$ and $\xi_2 = \xi_2^*$.
Hence $(\xi_1, \xi_2) = (\xi_1^*, \xi_2^*)$, proving that $(\xi_1, \xi_2)$is the one and only one coupled fixed point of μ in E. □

Example 3.1 *Let E = [0, 1]. We know that* $\mathrm{G_{JS}} : \mathrm{E}^3 \to [0, \infty]$ *defined by,*
$\mathrm{G_{JS}}(\xi, \psi, \chi) = |\xi - \psi| + |\psi - \chi| + |\xi - \chi|$ *for all* $\chi, \psi, \xi \in \mathrm{E}$ *is a* $\mathrm{G_{JS}}$*- metric on E.*
Taking $\mu(\chi, \psi) = \frac{(\chi + \psi)^2}{2}$ *and* $\lambda = \frac{1}{2}$*, we can easily check that inequality* (3.1) *holds and every* $(\chi, \chi) \in \mathrm{E}^2$ *is a coupled fixed point of* μ *in E.*

4. Conclusion

In this article, we established a theorem which states that a map admits a coupled fixed point in $\mathrm{G_{JS}} - \textit{metric space}$ which is $\mathrm{G_{JS}}$- complete if it satisfies certain conditions. We also proved such a coupled fixed point is unique. A suitable example is provided to support the theorem.

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