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A Graph-Based Framework for Recursive Self-Invertible Matrix Cryptography Using Complete Graphs (K_4)

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ABSTRACT: This paper proposes a novel symmetric encryption scheme that integrates algebraic graph theory with recursive matrix transformations. The method encodes plaintext into weighted complete graphs (K_4) and encrypts the resulting adjacency matrices using a dynamic sequence of self-invertible keys. Unlike static Hill ciphers, this approach utilizes a recursive function to generate unique keys for each data block, significantly enhancing security against cryptanalysis. We define the algebraic structure over a modular arithmetic system (mod 29), detail the recursive key generation algorithm, and provide three distinct numerical illustrations of the encryption and decryption protocols.

Keywords: Cryptography, algebraic graph theory, Self-Invertible Matrix, recursive encryption, complete graphs.

Contents

1. Introduction

Cryptography relies heavily on matrix transformations to obscure relationships between plaintext and ciphertext [2,4,15]. While classical methods like the Hill Cipher utilize static keys, they are vulnerable to known-plaintext attacks due to the linear nature of the transformation. Recent research has explored using graph theory—specifically labeling and adjacency matrices—to introduce non-linearity and complexity [5,6,10,11].

This work extends previous research on self-invertible matrix cryptosystems [1,7,10,12,13] by introducing a Recursive Self-Invertible Matrix protocol. In this system, the encryption key evolves for every block of text based on a deterministic recursive function derived from a seed matrix. This ensures that identical plaintext blocks result in different ciphertext blocks, providing a robust defense against frequency analysis.

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2. Basic Graph Theory Concepts

To understand this method, we need to define a few simple terms [8,3,14,9]:

- **Graph:** A collection of dots (vertices) connected by lines (edges).
- **Complete Graph:** A graph where every dot is connected to every other dot directly.
- **Adjacency Matrix:** A grid of numbers representing the graph. If dot 1 is connected to dot 2, we put a number (weight) in the grid. If they are not connected, we put 0.

3. Preliminaries and Symbolic Framework

3.1. Encoding Scheme

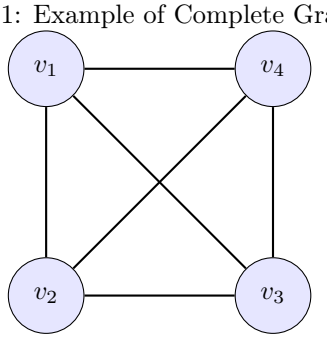
We utilize a modulo 29 arithmetic system to map characters to numerical values. The encoding table is defined as follows:

- Alphabets A through Z map to integers 1 through 26.
- The Space character maps to 27.
- The Dummy character '&' maps to 28.
- All arithmetic operations are performed modulo 29.

3.2. Complete Graph K_4 Representation

Let a message block consist of 6 characters. These characters are mapped to the 6 edges of a complete graph K_4 with vertices $\{v_1, v_2, v_3, v_4\}$.

Figure 1: Example of Complete Graph K_4



Let the numerical values of the block be $\{w_1, w_2, w_3, w_4, w_5, w_6\}$. The mapping to edges is defined as:

$$(v_1, v_2) \to w_1 \quad (v_1, v_3) \to w_2 \quad (v_1, v_4) \to w_3$$
$$(v_2, v_3) \to w_4 \quad (v_2, v_4) \to w_5 \quad (v_3, v_4) \to w_6$$

The adjacency matrix A of this weighted graph is a symmetric 4×4 matrix where $A_{ij} = A_{ji}$ is the weight of the edge connecting v_i and v_j.

3.3. Self-Invertible Matrix Construction

A matrix K is self-invertible (or an involutory matrix) if $K^2 \equiv I \pmod{n}$. To create a large key matrix of size $n \times n$, we only need to create a smaller random matrix of size $\frac{n}{2} \times \frac{n}{2}$ [1]. Where I is an Identity matrix.
In this paper we construct a 4×4 self-invertible matrix K_{44} from a given 2×2 seed matrix K_{22}, using the block construction method:

$$K_{44} = \begin{bmatrix} -K_{22} & I - K_{22} \\ I + K_{22} & K_{22} \end{bmatrix} \pmod{29} \tag{3.1}$$

It is easily verified that $K_{44}^2 = I$, regardless of the choice of K_{22}.

4. Proposed Algorithm

4.1. Recursive Key Generation

To ensure dynamic encryption, the seed matrix K_{22} changes for each block i. Let the initial seed be $K_{22_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The recursive update rule for the i-th block (where $i \geq 0$) is defined as:

$$K_{22_i} = \begin{pmatrix} a+i & b-i \\ c-i & d+i \end{pmatrix} \pmod{29} \tag{4.1}$$

For each block i, a unique self-invertible key K_{44_i} is generated using K_{22_i} and Eq. (3.1).

4.2. Encryption Process

1. **Block Division:** Divide plaintext into blocks of 6 characters. If the last block has fewer than 6 characters, pad with '&' (value 28).
2. **Graph Conversion:** Convert each block i into weights and construct the adjacency matrix A_i.
3. **Key Generation:** For each block i, compute K_{22_i} and the corresponding K_{44_i}.
4. **Transformation:** Compute the cipher matrix C_i:

$$C_i = K_{44_i} \times A_i \times K_{44_i} \pmod{29}$$

5. **String Generation:** Flatten C_i (row-wise) into a string of numbers.
6. **Transmission:** The final output format is:

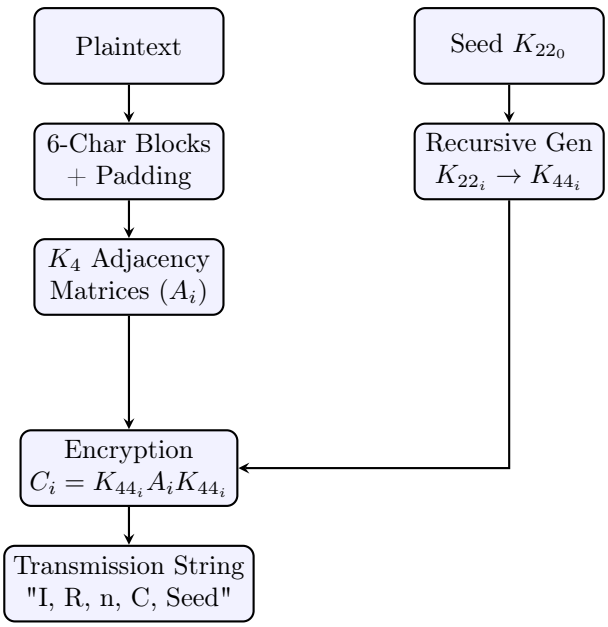
"I, R, n, < Encrypted Strings >, < Seed Elements >"

Where I is the number of blocks, $R = 1$ (indicating recursive mode) & $R = 0$ (indicating non-recursive mode), and $n = 4$ (matrix size).

4.3. Implementation Data Flow

The final transmission string for any message follows the structure defined as follows

Figure 2: Encryption Data Flow



4.4. Decryption Process

Decryption is the inverse of encryption. However, since the key matrices are self-invertible, the inverse key is the key itself ($K^{-1} = K$).

1. **Receive Data:** Extract the seed matrix K_{22_0} and the cipher matrices $C_0, C_1, ..., C_n$.
2. **Regenerate Keys:** Using the single shared seed K_{22_0}, the receiver independently calculates the sequence $K_{22_1}, K_{22_2}, ...$ using Eq. (4.1).
3. **Inverse Transformation:** For each block i:
$$A_i = K_{44_i} \times C_i \times K_{44_i} \pmod{29}$$
4. **Decoding:** Extract edges from the upper triangle of A_i to retrieve weights, convert to characters, and discard padding.

5. Worked Illustrations

Example 5.1 Example 1 :Single Block Illustration:
Plaintext: "GRAPH " (Note: Space at end)
Seed K_{22_0} : $\begin{pmatrix} 25 & 15 \\ 19 & 10 \end{pmatrix}$
Encryption:

1. **Encoding:** $G = 7$, $R = 18$, $A = 1$, $P = 16$, $H = 8$, $Space = 27$.
2. **Adjacency Matrix** (A_0)**:** Constructed from weights $\{7, 18, 1, 16, 8, 27\}$.
3. **Key** (K_{44_0})**:** Generated from K_{22_0}.
4. **Cipher** (C_0)**:** $C_0 = K_{44_0} A_0 K_{44_0} \pmod{29}$.

Decryption: The receiver possesses C_0 and the seed K_{22_0}.

1. **Key Regeneration:** The receiver computes K_{44_0} using the known seed.
2. **Matrix Recovery:** $A_0 = K_{44_0} C_0 K_{44_0} \pmod{29}$. Since K_{44_0} is self-invertible, this operation reverses the encryption perfectly.
3. **Extraction:** The receiver reads the upper triangle of A_0: Values $\{7, 18, 1, 16, 8, 27\} \rightarrow$ "GRAPH ".

Example 5.2 Example 2 :Three-Block Scenario (Comprehensive):
Plaintext: "SECURE DATA SET" (15 characters)
Structure:

- Block 0: "SECURE" (6 chars)
- Block 1: " DATA " (6 chars, includes spaces)
- Block 2: "SET" (3 chars) $\rightarrow$ Pads to "SET&&&"

Seed K_{22_0} : $\begin{pmatrix} 25 & 15 \\ 19 & 10 \end{pmatrix}$
Phase 1: Recursive Key Generation

- **Block 0 Key** $(i = 0)$**:** Generates K_{44_0} from K_{22_0}.
- **Block 1 Key** $(i = 1)$**:** Apply recursion: $a + 1$, $b - 1$, $c - 1$, $d + 1$.

$$K_{22_1} = \begin{pmatrix} 25+1 & 15-1 \\ 19-1 & 10+1 \end{pmatrix} = \begin{pmatrix} 26 & 14 \\ 18 & 11 \end{pmatrix} \pmod{29}$$

  Generates K_{44_1}.

- **Block 2 Key** $(i = 2)$**:** Apply recursion: $a + 2$, $b - 2$, $c - 2$, $d + 2$.

$$K_{22_2} = \begin{pmatrix} 25+2 & 15-2 \\ 19-2 & 10+2 \end{pmatrix} = \begin{pmatrix} 27 & 13 \\ 17 & 12 \end{pmatrix} \pmod{29}$$

  Generates K_{44_2}.

Phase 2: Encryption (Sender)

1. Block 0 ("SECURE"): Compute A_0. Encrypt: $C_0 = K_{44_0} A_0 K_{44_0}$.
2. Block 1 (" DATA "): Compute A_1. Encrypt: $C_1 = K_{44_1} A_1 K_{44_1}$.
3. Block 2 ("SET&&&"): Compute A_2. Encrypt: $C_2 = K_{44_2} A_2 K_{44_2}$.

Transmission String: "3, 1, 4, Flat C_0, Flat C_1, Flat C_2, 25, 15, 19, 10".
Phase 3: Decryption (Receiver) The receiver extracts the seed "25, 15, 19, 10".

1. **Block 0 Decryption:** Use seed to form K_{44_0}. Calculate $A_0 = K_{44_0} C_0 K_{44_0} \pmod{29}$. Extract weights $\rightarrow$ "S, E, C, U, R, E".
2. **Block 1 Decryption:** Compute K_{22_1} by adding index 1 to diagonals and subtracting 1 from off-diagonals of the seed. Form K_{44_1}. Calculate A_1 and extract weights $\rightarrow$ " , D, A, T, A, ".
3. **Block 2 Decryption:** Compute K_{22_2} by adding index 2 to diagonals and subtracting 2 from off-diagonals of the seed. Form K_{44_2}. Calculate A_2. Extract weights "S, E, T, &, &, &". Post-processing: The receiver identifies '&' (28) as dummy characters and removes them.

Final Result: "SECURE DATA SET".

Example 5.3 Example 3 : Padding Detail:
Plaintext: "KEY" (3 chars)
Process: Since the text is less than 6 characters, it is treated as a single block requiring padding.

- Block: "KEY&&&".
- Values: $K(11)$, $E(5)$, $Y(25)$, &(28), &(28), &(28).
- Adjacency Matrix (A_0):

$$A_0 = \begin{pmatrix} 0 & 11 & 5 & 25 \\ 11 & 0 & 28 & 28 \\ 5 & 28 & 0 & 28 \\ 25 & 28 & 28 & 0 \end{pmatrix}$$

- Encryption/Decryption: Follows the standard single-block procedure ($i = 0$).

6. Conclusion

This paper presented a recursive matrix cryptosystem based on complete graph adjacency matrices. By integrating a time-varying key generation mechanism (K_{22_i}), the system overcomes the static key vulnerabilities of traditional algebraic ciphers. The use of self-invertible matrices simplifies the protocol by eliminating the need for matrix inversion during decryption, making it computationally efficient for both sender and receiver.

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