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Computational Method for Singularly Perturbed Differential-Difference Equation via Mixed Non-Polynomial Quadratic Spline Approach

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ABSTRACT: In this paper, a mixed nonpolynomial quadratic spline technique is implemented using three nodal points for solving singularly perturbed differential-difference equation which is reduced to an equivalent two point singularly perturbed boundary value problem. A fitting factor is incorporated in the second order finite difference scheme which handles the oscillations that arising in the boundary layer. The discrete system obtained from the difference scheme is solved by employing Thomas algorithm. A brief convergence analysis is demonstrating that the suggested approach achieves fourth-order convergence. Numerous illustrations are provided to highlight the efficiency of the suggested approach. Comparative analysis is made to validate the accuracy and reliability of the proposed strategy.

Keywords: Singularly perturbed differential-difference equation, non-polynomial quadratic spline, boundary value problem.

Contents

1. Introduction

Difference equations in which the leading derivative term is multiplied by a small positive parameter and involving at least one delay or advance parameter or both referred to as Singularly perturbed differential difference equations (SPDDEs). These differential equations are widely implemented in mathematical modelling to accurately predict real-world phenomena across many scientific and engineering domains. Such differential equations plays a significant role in the fields of biosciences, in the study of human pupil light reflex, control theory, economics, and engineering. Numerous strategies have been incorporated to handle the difficulties arising in SPDDEs in recent years. Such problems frequently encountered in Control analysis and design studies [10], hybrid optical systems [2], in the study of population dynamics [11], physiological control system investigations [16] and competitive tumour growth models [25]. For a more detailed understanding of the mathematical models in this domain, the equations are organized by following the formulations presented in Doolan and Miller [3] and Kokotovic et al. [10] based on these models.

The authors in [1] derived two parameter fitted scheme to get the accurate solution and tackled boundary layer behaviour by introducing two fitting factors at convective and diffusion terms. An exponentially fitted spline method is presented in [23]. The authors in [5] developed B-spline collocation approach with

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fitted mesh to generate a piecewise-uniform mesh. To obtain the standard finite difference scheme the authors in [8] derived an ϵ-uniform convergent scheme by introducing an exponential fitting parameter. The authors in [6] constructed a parameter uniform difference scheme to solve the BVP by employing Taylor approximation to handle the terms containing small shifts. Kadalbajoo et al. [7] presented an ε-uniform fitted operator method for SPDDEs containing small delay with boundary layer behaviour. In [22], Galerkin approach with fitting factor is proposed. The authors in [9] proposed an ε-uniform fitted mesh method to solve SPDDEs containing negative as well as positive shifts with layer behaviour. In [12] authors proposed a mixed finite difference method with mixed shifts. Lange & Miura [13–15] implemented fundamental research on BVPs associated with expected first-exit times of the membrane potential in models for neurons. The authors concentrated on problems whose solutions exhibits layer behaviour at one or both of the boundaries. The analysis of the layer equations are analysed by employing Laplace transforms which leads to significant results. The authors in [21] proposed a computational approach for SPDDEs by reducing second-order problem to an asymptotically equivalent first-order neutral type delay differential equation, which is then discretized using numerical integration and linear interpolation to obtain a tridiagonal system. To acquire more comprehensive understanding of SPDDEs, readers are suggested to refer the publications [4,18,20] and the references cited therein.

2. Description of the Problem

Let us consider a class of singularly perturbed differential–difference equation of the form

$$\varepsilon\theta''(s) + p(s)\theta'(s) + q(s)\theta(s-\delta) + r(s)\theta(s) + \tau(s)\theta(s+\eta) = f(s) \tag{1}$$

over the domain (0, 1), subject to the boundary conditions

$$\begin{aligned} \theta(s) &= \phi(s), \quad -\delta \le s \le 0,\\ \theta(s) &= \vartheta(s), \quad 1 \le s \le 1+\eta. \end{aligned} \tag{2}$$

Here $0 < \varepsilon \ll 1$ denotes a small perturbation parameter, the coefficient functions $p(s), q(s), r(s), \tau(s), f(s)$, $\phi(s)$ and $\vartheta(s)$ are assumed to be sufficiently smooth and $0 < \delta = o(\varepsilon)$, $0 < \eta = o(\varepsilon)$ are respectively the delay (negative shift) and the advance (positive shift) parameters. The solution behaviour of Eq. (1) together with the boundary conditions in Eq. (2) depends on the sign of $p(s) - \delta q(s) + \eta\tau(s)$. specifically, layer exists at the left-end of the domain when this expression is positive and layer exists at the right-end of the domain when this expression is negative. If $p(s) = 0$, then problem may exhibit either oscillatory solution or two layers based on the case whether $q(s) + r(s) + \tau(s)$ takes positive or negative.

Since the solution $\theta(s)$ of the Eq. (1) is sufficiently smooth, the terms $\theta(s-\delta)$ and $\theta(s-\eta)$ can be approximated using Taylor series expansion, leading to the following expressions

$$\theta(s-\delta) \approx \theta(s) - \delta\theta'(s) \tag{3a}$$

$$\theta(s+\eta) \approx \theta(s) + \eta\theta'(s) \tag{3b}$$

Employing Eq. (3) in Eq. (1), we obtain

$$\varepsilon\theta''(s) + u(s)\theta'(s) + v(s)\theta(s) = f(s) \quad \text{with boundary conditions } \theta(a) = \alpha,\ \theta(b) = \beta \tag{4}$$

Here, $u(s) = p(s) - \delta q(s) + \eta\tau(s),\ v(s) = q(s) + r(s) + \tau(s).$ (5)

Eq. (4) represents a second order singular perturbation problem involving a small perturbation parameter $0 < \varepsilon \ll 1$. Assume that the coefficient functions $u(s), v(s), f(s)$ are sufficiently smooth over the domain $[a, b]$, while α, β represents finite constants. The solution exhibits boundary layer in the neighbourhood of $s = a$, provided that $u(s) \ge L > 0$ across the entire domain $[a, b]$, where L is positive constant. On the other hand if there is a negative constant $\bar{L}$ such that $u(s) \le \bar{L} < 0$ over the domain $[a, b]$, then the solution exhibits boundary layer in the neighbourhood of $s = b$.

3. Mixed Non Polynomial Quadratic Spline

Let $a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b$, the interval is subdivided into n uniform subintervals by defining grid points $s_i = a + ih, i = 0, 1, \ldots, n$ and $h = (b-a)/n$.

For each segment $[s_i, s_{i+1}]$, $i = 0, 1, \ldots, n-1$, the mixed non- polynomial quadratic spline of the form

$$\psi_i(s) = a_i e^{k(s-s_i)} + b_i\left[\cos k(s-s_i) + \sin k(s-s_i)\right] + c_i, \qquad i = 0, 1, 2, \ldots, n \tag{6}$$

defined on $[a, b]$ is constructed and this spline interpolates the grid points s_i depends on a parameter k. In the limiting case $k \to 0$ the proposed spline reduces to an ordinary quadratic spline over the domain $[a, b]$ and unknown coefficients a_i, b_i and c_i are determined by employing the following interpolatory conditions

$$\psi_i(s_i) = \theta_i, \quad \psi_i(s_{i+1}) = \theta_{i+1},$$

$$\psi_i''(s_i) = \tfrac{1}{2}(M_i + M_{i+1}), \quad i = 0, 1, \ldots, n \tag{7}$$

By employing constraints in Eq. (7) we derived the coefficients as

$$a_i = \frac{\theta_{i+1} - \theta_i}{\sin\omega + \cos\omega + e^\omega - 2} - \frac{h^2}{2\omega^2}\left[\frac{e^\omega - 1}{\sin\omega + \cos\omega + e^\omega - 2} - 1\right](M_i + M_{i+1}),$$

$$b_i = \frac{\theta_{i+1} - \theta_i}{\sin\omega + \cos\omega + e^\omega - 2} - \frac{h^2}{2\omega^2}\left[\frac{e^\omega - 1}{\sin\omega + \cos\omega + e^\omega - 2}\right](M_i + M_{i+1}),$$

$$c_i = \frac{-2(\theta_{i+1} + \theta_i)(\sin\omega + \cos\omega + e^\omega)}{\sin\omega + \cos\omega + e^\omega - 2} + \frac{h^2}{2\omega^2}\left[\frac{2(e^\omega - 1)}{\sin\omega + \cos\omega + e^\omega - 2} - 1\right](M_i + M_{i+1}).$$

Where $\omega = kh$.

By employing the continuity of first derivative at grid point $s = s_i$ namely $\psi_{i-1}'(s_i) = \psi_i'(s_i)$, produces the following tridiagonal system

$$\alpha\theta_{i-1} + \beta\theta_i + \gamma\theta_{i+1} = h^2\left(\alpha_1 M_{i-1} + \beta_1 M_i + \gamma_1 M_{i+1}\right), \quad i = 1, 2, \ldots, n-1 \tag{8}$$

Where

$$\alpha = \frac{e^\omega + \cos\omega - \sin\omega}{2}, \quad \beta = \frac{\sin\omega - \cos\omega - e^\omega - 2}{2}, \quad \gamma = 1,$$

$$\alpha_1 = \frac{(2\sin\omega - 1)e^\omega + \cos\omega - \sin\omega}{4\omega^2}, \quad \beta_1 = \frac{e^\omega \sin\omega - \sin\omega}{2\omega^2}, \quad \gamma_1 = \frac{e^\omega - \sin\omega - \cos\omega}{4\omega^2}.$$

$$\omega \to 0, \text{ then } (\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1) \to \left(1, -2, 1, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right).$$

As a result, the spline defined by Eq. (8) degenerates into an ordinary cubic spline relation as described below

$$\theta_{i-1} - 2\theta_i + \theta_{i+1} = \frac{h^2}{4}\left(M_{i-1} + 2M_i + M_{i+1}\right) \tag{9}$$

The relation Eq. (8) gives $(n-1)$ linear algebraic equations in $(n-1)$ unknowns $s_i, i = 1, 2, \ldots, n-1$.

4. Derivation of the Difference Scheme

Discretization of the suggested SPDDE represented with Eq. (1) at the grid point s_i yields the following equation

$$\varepsilon\theta''(s_i) + u(s_i)\theta'(s_i) + v(s_i)\theta(s_i) = f(s_i) \tag{10}$$

By incorporating second derivative of spline, the following expressions are obtained

$$M_i = \frac{f_i - u_i\theta_i' - v_i\theta_i}{\varepsilon}, \quad M_{i-1} = \frac{f_{i-1} - u_{i-1}\theta_{i-1}' - v_{i-1}\theta_{i-1}}{\varepsilon}, \quad M_{i+1} = \frac{f_{i+1} - u_{i+1}\theta_{i+1}' - v_{i+1}\theta_{i+1}}{\varepsilon}$$

Where

$$\theta'_i = \frac{\theta_{i+1} - \theta_{i-1}}{2h}, \quad \theta'_{i-1} = \frac{-\theta_{i+1} + 4\theta_i - 3\theta_{i-1}}{2h}, \quad \theta'_{i+1} = \frac{3\theta_{i+1} - 4\theta_i + \theta_{i-1}}{2h}$$

$$u_i = u(s_i), \quad v_i = v(s_i), \quad f_i = f(s_i)$$

Employing the values of M_j $(j = i, i \pm 1)$ in Eq. (8), we obtain

$$\begin{aligned}
&\left[\alpha\varepsilon - \tfrac{3}{2}\alpha_1 u_{i-1}h - \tfrac{\beta_1}{2}u_i h + \tfrac{\gamma_1}{2}u_{i+1}h + \alpha_1 v_{i-1}h^2\right]\theta_{i-1} \\
&+ \left[\beta\varepsilon + 2\alpha_1 u_{i-1}h + \beta_1 v_i h^2 - 2\gamma_1 u_{i+1}h\right]\theta_i \\
&+ \left[\gamma\varepsilon - \tfrac{1}{2}\alpha_1 u_{i-1}h + \tfrac{\beta_1}{2}u_i h + \tfrac{3}{2}\gamma_1 u_{i+1}h + \gamma_1 v_{i+1}h^2\right]\theta_{i+1} \\
&= h^2\left[\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right]
\end{aligned} \tag{11}$$

we get the following system

$$L_i\theta_{i-1} + C_i\theta_i + U_i\theta_{i+1} = H_i \quad \text{for } i = 1, 2, \dots, N-1 \tag{12}$$

Where

$$\begin{aligned}
L_i &= \alpha\varepsilon - \tfrac{3}{2}\alpha_1 u_{i-1}h - \tfrac{\beta_1}{2}u_i h + \tfrac{\gamma_1}{2}u_{i+1}h + \alpha_1 v_{i-1}h^2, \\
C_i &= \beta\varepsilon + 2\alpha_1 u_{i-1}h + \beta_1 v_i h^2 - 2\gamma_1 u_{i+1}h, \\
U_i &= \gamma\varepsilon - \tfrac{1}{2}\alpha_1 u_{i-1}h + \tfrac{\beta_1}{2}u_i h + \tfrac{3}{2}\gamma_1 u_{i+1}h + \gamma_1 v_{i+1}h^2, \\
H_i &= h^2\left[\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right].
\end{aligned}$$

The tridiagonal system Eq. (12) is solved for $i = 1, 2, \dots, N-1$ to obtain the approximations $\theta_1, \theta_2, \dots, \theta_{N-1}$ of the solution $\theta(s)$ at $s_1, s_2, \dots, s_{N-1}$.

Now introduce a fitting factor $\sigma(\rho)$ in Eq. (10)

$$\left[\frac{\sigma_i}{\rho} - \frac{3}{2}\alpha u_{i-1} - \frac{u_i}{2}\beta + \frac{\alpha}{2}u_{i+1}\right]\theta_{i-1} + \left[\frac{-2\sigma_i}{\rho} + 2\alpha u_{i-1} - 2\alpha u_{i+1}\right]\theta_i + \left[\frac{\sigma_i}{\rho} + \frac{3}{2}\alpha u_{i-1} + \frac{u_i}{2}\beta - \frac{\alpha}{2}u_{i+1}\right]\theta_{i+1} = 0$$

$$\frac{\sigma_i}{\rho}\left(\theta_{i-1} - 2\theta_i + \theta_{i+1}\right) + \left(-\alpha a_i - \tfrac{\beta a_i}{2}\right)\theta_{i-1} + \left(\alpha a_i + \tfrac{\beta a_i}{2}\right)\theta_{i+1} = 0$$

The value of $\sigma_i(\rho)$ is acquired by the procedure given by Doolan et al. [3] and is

$$\sigma_i(\rho) = \frac{\rho a_i}{2}\coth\left(\frac{a_i\rho}{2}\right)$$

5. Truncation Error

The local truncation error developed for the proposed scheme represented in Eq. (12) as follows

$$\begin{aligned}
T_i(h) = {}&(\alpha + \beta + \gamma)\varepsilon + (\gamma - \alpha)\varepsilon\theta'_i h + \left[\left(\frac{\alpha + \gamma}{2}\right) - (\alpha_1 + \beta_1 + \gamma_1)\right]\varepsilon\theta''_i h^2 \\
&+ \left[\left(\frac{\gamma - \alpha}{6}\right) - (\gamma_1 - \alpha_1)\right]\varepsilon\theta'''_i h^3 + \left[\left(\frac{\alpha + \gamma}{4!}\right)\varepsilon\theta^{iv}_i + \left(\frac{\alpha_1 + \beta_1 + \gamma_1}{3!}\right)u_i\theta'''_i \right. \\
&\left. - \left(\frac{\alpha_1 + \gamma_1}{2}\right)\left(\varepsilon\theta^{iv}_i + u_i\theta'''_i\right)\right]h^4 \dots
\end{aligned}$$

Thus, truncation error demonstrates that the numerical scheme achieves fourth order for various values of $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$.

Remarks:

For $\alpha = 1, \beta = -2, \gamma = 1$ and $\alpha_1 = \frac{1}{4}, \beta_1 = \frac{1}{2}, \gamma_1 = \frac{1}{4}$ truncation error is of fourth order.

6. Convergence Analysis

The convergence analysis of the proposed approach to the Eq. (1) is now being investigated. The corresponding system of equations are represented in matrix form with the boundary conditions is obtained as

$$(D+F)W+G+T(h)=0 \tag{13}$$

where

$$D=[\varepsilon,\ \varepsilon\gamma,\ \varepsilon]=\begin{bmatrix}\varepsilon\gamma & \varepsilon & 0 & 0 & \cdots & 0\\ \varepsilon & \varepsilon\gamma & \varepsilon & 0 & \cdots & 0\\ 0 & \varepsilon & \varepsilon\gamma & \varepsilon & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & 0 & \varepsilon & \varepsilon\gamma\end{bmatrix}$$

and

$$F=[\tilde{z}_i,\tilde{r}_i,\tilde{w}_i]=\begin{bmatrix}\tilde{r}_1 & \tilde{w}_1 & 0 & 0 & \cdots & 0\\ \tilde{z}_2 & \tilde{r}_2 & \tilde{w}_2 & 0 & \cdots & 0\\ 0 & \tilde{z}_3 & \tilde{r}_3 & \tilde{w}_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & 0 & \tilde{z}_{N-1} & \tilde{r}_{N-1}\end{bmatrix}$$

$$\tilde{z}_i=-\frac{3\alpha_1}{2}u_{i-1}h-\frac{\beta_1}{2}u_ih+\beta_1v_i+\frac{\gamma_1}{2}u_{i+1}h+\alpha v_{i-1}h^2$$

$$\tilde{r}_i=2\alpha_1u_{i-1}h+\beta_1v_ih^2-2\gamma_1u_{i+1}h$$

$$\tilde{w}_i=-\frac{\alpha_1}{2}u_{i-1}h+\frac{\beta_1}{2}u_ih+\frac{3\gamma_1}{2}u_{i+1}h+\gamma_1v_{i+1}h^2 \qquad \text{for } i=1,2,\ldots,N-1$$

and

$$G=[\,q_1-\tilde{z}_1\alpha,\ q_2,\ q_3,\ \ldots,\ q_{N-1}-\tilde{w}_{N-1}\beta\,]$$

where

$$q_i=h^2\left[\alpha_1f_{i-1}+\beta_1f_i+\gamma_1f_{i+1}\right], \qquad \text{for } i=2,3,\ldots,N-1$$

$$T(h)=O(h^4) \quad \text{for } \alpha=1,\beta=-2,\gamma=1 \text{ and } \alpha_1=\tfrac{1}{4},\beta_1=\tfrac{1}{2},\gamma_1=\tfrac{1}{4}$$

and

$$W=[W_1,W_2,W_3,\ldots,W_{N-1}]^T, \quad T(h)=[T_1,T_2,\ldots,T_{N-1}]^T, \quad O=[0,0,\ldots,0]^T$$

are associated vectors of equation (10).
Let $w=[w_1,w_2,\ldots,w_{N-1}]^T\in\mathcal{W}$, which satisfies the equation

$$(D+F)w+G=O$$

Let $e_i=w_i-W_i$, $i=1,2,3,4,\ldots,N-1$ be the discretization error so that

$$E=[e_1,e_2,\ldots,e_{N-1}]^T=w-W.$$

Subtracting Eq. (13) from Eq. (14), we obtain the error equation

$$(D+F)E=T(h) \tag{14}$$

Let $|u(s)| \leq \xi_1$ and $|v(s)| \leq \xi_2$ where ξ_1, ξ_2 are positive constants. If $F_{(i,j)}$ be the $(i,j)^{\text{th}}$ element of F, then

$$|u_{i,i+1}| = |\tilde{w}_i| \leq \sigma_i\varepsilon - \frac{\alpha_1}{2}u_{i-1}h + \frac{\beta_1}{2}u_i h + \frac{3\gamma_1}{2}u_{i+1}h + \gamma_1 v_{i+1}h^2, \quad \text{for } i = 1, 2, \ldots, N-2$$

$$|u_{i,i-1}| = |\tilde{z}_i| \leq \sigma_i\varepsilon - \frac{3\alpha_1}{2}u_{i-1}h - \frac{\beta_1}{2}u_i h + \beta_1 v_i + \frac{\gamma_1}{2}u_{i+1}h + \alpha v_{i-1}h^2, \quad \text{for } i = 2, 3, \ldots, N-1$$

Thus, for sufficiently small h_i, we have

$$|u_{i,i+1}| < \varepsilon, \quad i = 1, 2, \ldots, N-2 \tag{15a}$$

$$|u_{i,i-1}| < \varepsilon, \quad i = 2, 3, \ldots, N-2 \tag{15b}$$

Hence $(D+F)$ is irreducible (Ref. [26]).

Let the sum of the elements of the i^{th} row of the matrix $(D+F)$ be $\bar{S}_i$, then we acquire

$$\bar{S}_i = \sum_{j=1}^{N-1} M_{ij} = \frac{-\sigma_i\varepsilon}{h^2} - \frac{\alpha_1}{2h}u_{i-1} + \frac{\beta_1}{2h}u_i + \frac{3\gamma_1}{2h}u_{i+1} + \gamma_1 v_{i+1}, \quad \text{for } i = 1$$

$$\bar{S}_i = \sum_{j=1}^{N-1} M_{ij} = -\frac{2\alpha_1}{h}u_{i-1} - \frac{2\gamma_1}{h}u_{i+1} + \alpha_1 v_{i-1} + \gamma_1 v_{i+1}, \quad \text{for } i = 2, 3, \ldots, N-2$$

$$\bar{S}_{N-1} = \sum_{j=1}^{N-1} M_{(N-1)j} = \frac{-\sigma_i\varepsilon}{h^2} + \frac{\alpha_1}{2h}u_{i-1} - \frac{\beta_1}{2h}u_i + \beta_1 v_i - \frac{3\gamma_1}{2h}u_{i+1} + \alpha v_{i-1}, \quad \text{for } i = N-1$$

Let $\xi_{1^*} = \min\limits_{1\leq i\leq N} |u(s_i)|$ and $\xi_1^* = \max\limits_{1\leq i\leq N} |u(s_i)|$, $\xi_{2^*} = \min\limits_{1\leq i\leq N} |v(s_i)|$ and $\xi_2^* = \max\limits_{1\leq i\leq N} |v(s_i)|$.

Since $0 < \varepsilon \ll 1$ and $\varepsilon \propto O(h)$, it is verify that for sufficiently small h, $(D+F)$ is monotone [24,26]. Hence $(D+F)^{-1}$ exists and $(D+F)^{-1} \geq 0$.

Thus from Eq. (14), we get

$$\|E\| \leq \|(D+F)^{-1}\| \, \|T\| \tag{16}$$

Let $(D+F)^{-1}_{i,k}$ be the $(i,k)^{\text{th}}$ element of $(D+F)^{-1}$ and we define

$$\|(D+F)^{-1}\| = \max_{1\leq i\leq N-1} \sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \quad \textit{and} \quad \|T(h)\| = \max_{1\leq i\leq N-1} |T(h)| \tag{17a}$$

Since $(D+F)^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{N-1}\left[(D+F)^{-1}_{i,k}\bar{S}_k = 1\right]$ for $i = 1, 2, \ldots, N-1$. Hence

$$(D+F)^{-1}_{i,1} \leq \frac{1}{\bar{S}_1} < \frac{1}{h^2\xi} \tag{17b}$$

$$(D+F)^{-1}_{i,N-1} \leq \frac{1}{\bar{S}_{N-1}} < \frac{1}{h^2\xi} \tag{17c}$$

Furthermore,

$$\sum_{k=2}^{N-2} (D+F)^{-1}_{i,k} \leq \frac{1}{\min\limits_{2\leq k\leq N-2} \bar{S}_k} < \frac{1}{h^2\xi}, \quad i = 2, 3, \ldots, N-2.$$

By employing Eq. (16) with the help of Eqs. (17a)–(17d), the following expression is obtained

$$\|E\| \leq O(h^2) \tag{18}$$

Hence the method given in Eq. (13) is second order convergent for $\alpha = 1, \beta = -2, \gamma = 1$ and $\alpha_1 = \frac{1}{4}, \beta_1 = \frac{1}{2}, \gamma_1 = \frac{1}{4}$.

7. Computational Illustrations

In order to demonstrate the efficiency and applicability of the proposed approach, model problems governed by Eqs. (1)–(2) are considered whose exact solution is

$$\theta(s) = c_1 e^{m_1 s} + c_2 e^{m_2 s} + \frac{f}{C}$$

where

$$C = b + c + d, \quad c_1 = \frac{-f + yC + e^{m_2}(f - \phi C)}{(e^{m_1} - e^{m_2})C}, \quad c_2 = \frac{f - yC + e^{m_1}(-f + \phi C)}{(e^{m_1} - e^{m_2})C}.$$

$$m_1 = \frac{-(p - q\delta + \omega\eta) + \sqrt{(p - q\delta + \omega\eta)^2 - 4\varepsilon C}}{2\varepsilon}, \quad m_2 = \frac{-(p - q\delta + \omega\eta) - \sqrt{(p - q\delta + \omega\eta)^2 - 4\varepsilon C}}{2\varepsilon}.$$

Illustration 1. $\varepsilon\theta''(s) + \theta'(s) - 2\theta(s-\delta) - 5\theta(s) + \theta(s+\eta) = 0$ with $\theta(s) = 1$, $-\delta \le s \le 0$, $\theta(s) = 0$, $1 \le s \le 1+\eta$.

Illustration 2. $\varepsilon\theta''(s) - \theta'(s) - 2\theta(s-\delta) + \theta(s) - 2\theta(s+\eta) = 0$ with $\theta(s) = 1$, $-\delta \le s \le 0$, $\theta(s) = -1$, $1 \le s \le 1+\eta$.

Illustration 3. $\varepsilon\theta''(s) - (1 + e^{s^2})\theta'(s) - se^s\theta(s-\delta) - s^2\theta(s) - (1 - e^{-s})\theta(s+\eta) = 1$ with $\theta(s) = 1$, $-\delta \le s \le 0$, $\theta(s) = -1$, $1 \le s \le 1+\eta$.

Illustration 4. $\varepsilon\theta''(s) + (1 + e^{-s^2})\theta'(s) + s\theta(s-\delta) + s^2\theta(s) + (1.5 - e^{-s})\theta(s+\eta) = -1$ with $\theta(s) = 1$, $-\delta \le s \le 0$, $\theta(s) = 1$, $1 \le s \le 1+\eta$.

8. Discussions and Conclusion

This article presents a numerical technique employing a mixed non polynomial quadratic spline for solving the SPDDE. A three-term relation is formulated using the difference approach. A brief analysis of the scheme's convergence is demonstrated. The efficiency of the proposed technique is presented through several computational illustrations. The proposed approach has been validated and shown to be effective in addressing numerous problems.

The MAEs for the solutions of illustrations 1–4 are listed in Tables 1–4. Graphical representation of the solutions for various values of δ, η are depicted at Figures 1–8. A comparison of the computed errors indicate that the proposed approach yields higher accuracy over the approach produced in [8,17]. Analysis of figures 1–8 indicates that, an increase in δ and η reduces the width of the left end boundary layer in figure 1–2 while it enlarges the width of the right end boundary layer in Figure 3–4. Additionally, an increase in δ enlarges the width of the right-end boundary layer in figure 5 and reduces the width of the left-end boundary layer in Figure 7. Conversely, increasing η reduces the width of the right-end boundary layer in figure 6 and enlarges the width of the left-end boundary layer figure 8.

Table 1: MAE corresponding to the solution of illustration 1.

$N \rightarrow$	16	32	64	128	256	512	1024
10^{0}	4.5074e-04	1.1193e-04	2.7936e-05	6.9811e-06	1.7454e-06	4.3634e-07	1.0908e-07
10^{-2}	2.3265e-02	1.1983e-02	4.3220e-03	1.0486e-03	2.5129e-04	6.3205e-05	1.5760e-05
10^{-4}	2.3405e-02	1.3362e-02	7.1987e-03	3.7459e-03	1.9121e-03	9.6619e-04	4.8562e-04
10^{-8}	2.3405e-02	1.3362e-02	7.1987e-03	3.7459e-03	1.9121e-03	9.6619e-04	4.8567e-04
10^{-10}	2.3405e-02	1.3362e-02	7.1987e-03	3.7459e-03	1.9121e-03	9.6619e-04	4.8567e-04
Results in [17]							
10^{0}	1.8954e-02	1.0127e-02	5.2399e-03	2.6665e-03	1.3450e-03	6.7547e-04	3.3848e-04
10^{-2}	9.4554e-02	5.9631e-02	3.4797e-02	1.9060e-02	9.9959e-03	5.0997e-03	2.5832e-03
10^{-4}	1.4746e-01	9.2545e-02	5.5074e-02	3.1902e-02	1.8077e-02	1.0086e-02	5.5420e-03
10^{-8}	1.5356e-01	9.6126e-02	5.7165e-02	3.3247e-02	1.8984e-02	1.0684e-02	5.9443e-03
10^{-10}	1.5356e-01	9.6126e-02	5.7165e-02	3.3247e-02	1.8984e-02	1.0684e-02	5.9443e-03

Table 2: MAE corresponding to the solution of illustration 2 for $\varepsilon = 0.1$ and for different values of η and δ.

$N \rightarrow$	8	32	128	512
$\delta \downarrow$ $\eta = 0.05$				
0.00	1.8115e-02	1.0328e-03	6.4031e-05	4.0020e-06
0.05	1.6810e-02	9.5187e-04	5.9260e-05	3.7026e-06
0.09	1.5665e-02	8.8187e-04	5.5198e-05	3.4484e-06
$\eta \downarrow$ $\delta = 0.05$				
0.00	1.5365e-02	8.6624e-04	5.4148e-05	3.3829e-06
0.05	1.6810e-02	9.5187e-04	5.9260e-05	3.7026e-06
0.09	1.7866e-02	1.0172e-03	6.3105e-05	3.9437e-06
Results in [8]				
$\delta \downarrow$ $\eta = 0.05$				
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow$ $\delta = 0.05$				
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

Table 3: MAE corresponding to the solution of illustration 3

ε/N	16	32	64	128	256	512	1024
10^0	2.9761e-05	8.9724e-06	2.4135e-06	6.1108e-07	1.5325e-07	3.8365e-08	9.5929e-09
10^{-2}	7.6939e-03	4.3737e-03	2.1386e-03	7.9044e-04	2.2838e-04	5.9487e-05	1.5031e-05
10^{-4}	7.6936e-03	4.3881e-03	2.3193e-03	1.1907e-03	6.0310e-04	3.0350e-04	1.5223e-04
10^{-8}	7.6936e-03	4.3881e-03	2.3193e-03	1.1907e-03	6.0310e-04	3.0350e-04	1.5223e-04
10^{-10}	7.6936e-03	4.3881e-03	2.3193e-03	1.1907e-03	6.0310e-04	3.0350e-04	1.5223e-04
Results in [17]							
10^0	2.5615e-02	1.3405e-02	6.8223e-03	3.3955e-03	1.6455e-03	7.6147e-04	3.1774e-04
10^{-2}	1.5346e-01	8.9743e-02	4.6893e-02	1.9818e-02	9.8145e-03	5.3983e-03	2.5123e-03
10^{-4}	8.2696e-02	5.6596e-02	3.7338e-02	2.4148e-02	1.3842e-02	7.5110e-03	3.5319e-03
10^{-8}	1.4572e-01	6.1728e-02	4.2696e-02	2.2476e-02	1.5602e-02	6.8759e-03	2.7374e-03
10^{-10}	1.0185e-01	6.5577e-02	3.8680e-02	2.2737e-02	9.3969e-03	4.9449e-03	2.2735e-03

Table 4: MAE corresponding to the solution of illustration 4 for $\delta = \eta = \varepsilon^2$

ε/N	32	64	128	256	512	1024
2^0	2.4949e-04	6.2483e-05	1.5616e-05	3.9040e-06	9.7601e-07	2.4400e-07
2^{-2}	3.0583e-03	7.7187e-04	1.9299e-04	4.8250e-05	1.2063e-05	3.0157e-06
2^{-4}	1.2278e-02	3.0840e-03	7.7445e-04	1.9417e-04	4.8548e-05	1.2138e-05
2^{-6}	2.6360e-02	1.0512e-02	2.9718e-03	7.5695e-04	1.9070e-04	4.7826e-05
2^{-8}	2.7047e-02	1.3402e-02	6.5460e-03	2.6276e-03	7.3860e-04	1.8967e-04
2^{-10}	2.7047e-02	1.3389e-02	6.6659e-03	3.3308e-03	1.6351e-03	6.5708e-04
2^{-12}	2.7047e-02	1.3389e-02	6.6658e-03	3.3265e-03	1.6618e-03	8.3166e-04
2^{-14}	2.7047e-02	1.3389e-02	6.6658e-03	3.3265e-03	1.6618e-03	8.3051e-04
2^{-16}	2.7047e-02	1.3389e-02	6.6658e-03	3.3265e-03	1.6618e-03	8.3051e-04
2^{-18}	2.7047e-02	1.3389e-02	6.6658e-03	3.3265e-03	1.6618e-03	8.3051e-04
2^{-20}	2.7047e-02	1.3389e-02	6.6658e-03	3.3265e-03	1.6618e-03	8.3051e-04

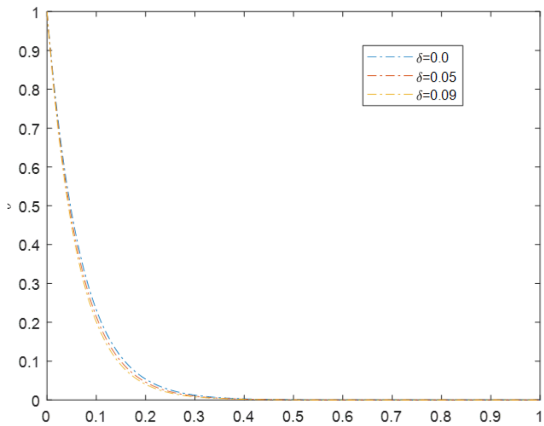

Fig 1: Visual interpretation of Illustration 1 with $\varepsilon = 0.1,\ \eta = 0.5\varepsilon$

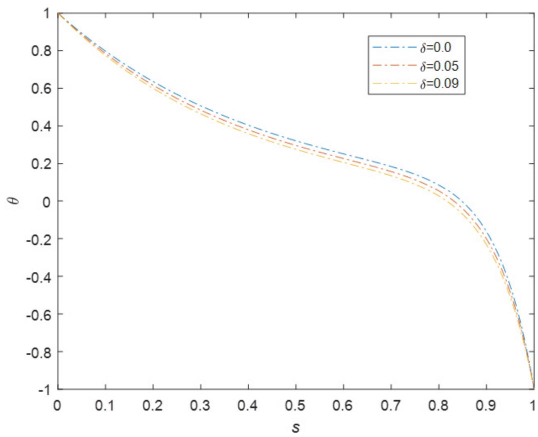

Fig 2: Visual interpretation of Illustration 2 with $\varepsilon = 0.1,\ \eta = 0.5\varepsilon$

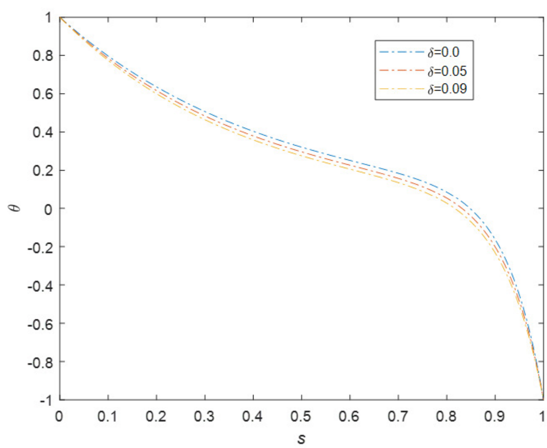

Fig 3: Visual interpretation of Illustration 2 with $\varepsilon = 0.1,\ \eta = 0.5\varepsilon$

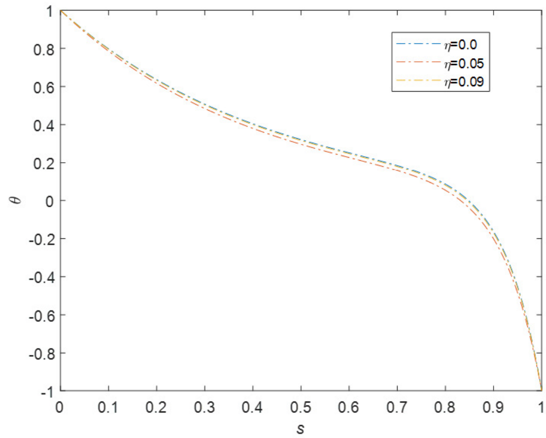

Fig 4: Visual interpretation of Illustration 2 with $\varepsilon = 0.1,\ \delta = 0.5\varepsilon$

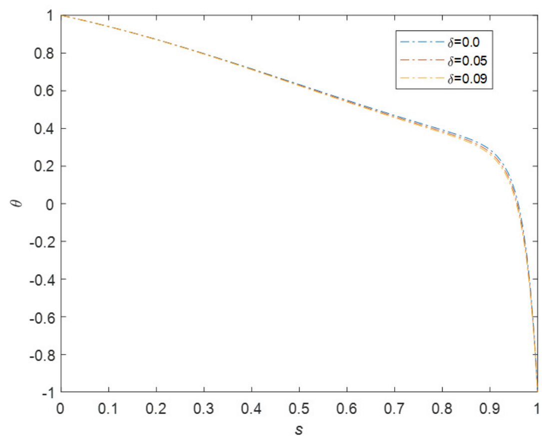

Fig 5: Visual interpretation of Illustration 3 with $\varepsilon = 0.1,\ \eta = 0.5\varepsilon$

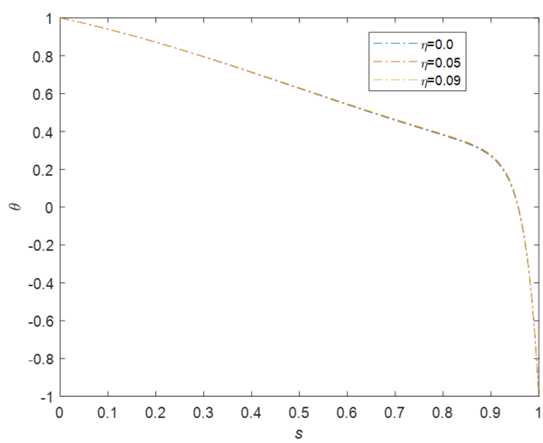

Fig 6: Visual interpretation of Illustration 3 with $\varepsilon = 0.1,\ \delta = 0.5\varepsilon$

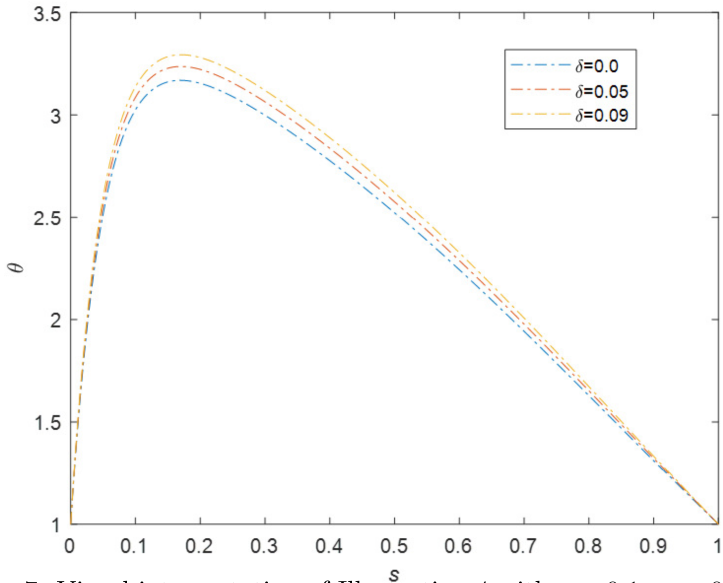

Fig 7: Visual interpretation of Illustration 4 with $\varepsilon = 0.1,\ \eta = 0.5\varepsilon$

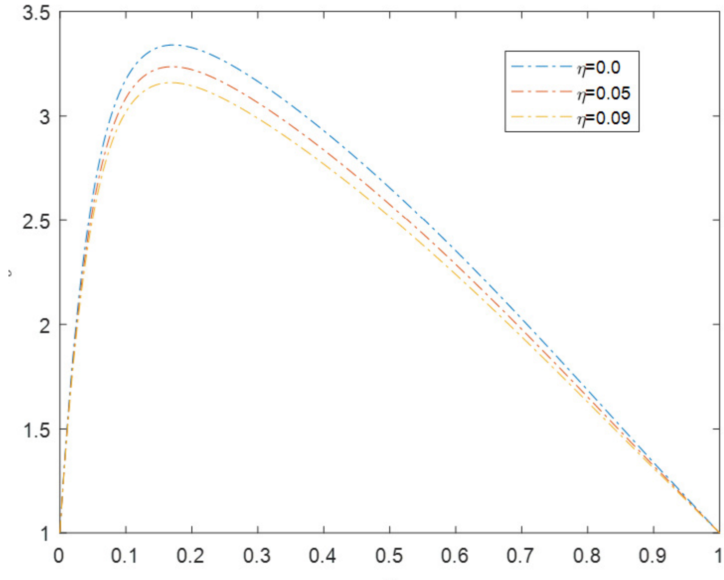

Fig 8: Visual interpretation of Illustration 4 with $\varepsilon = 0.1,\ \delta = 0.5\varepsilon$

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