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Rough Fuzzy Sub Near Algebras

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ABSTRACT: The present study merges rough sets and fuzzy sets to develop an innovative analytical framework in a near algebra. This manuscript explores the notion of rough fuzzy sub near algebra of a near algebra. We define this concept, establish key theorems, and investigate properties of homomorphisms on upper (lower) rough fuzzy sub near algebras (U(L)RFzSNAs).

Keywords: Rough set, fuzzy set, near algebra, upper (lower) rough fuzzy set.

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1. Introduction

Zadeh [20] pioneered fuzzy set theory (FST), which departed from classical set theory by introducing degree-based membership that allows elements to partially belong to a set. In addressing complex problems with incomplete data, FST has proven superior, providing a flexible and adaptive solution. FST has seen extensive development and global recognition, drawing attention from diverse fields. Rough set theory (RST), introduced by Pawlak [13], presented a unique mathematical framework for addressing uncertainty and vagueness in data. This approach diverges from traditional methods by focusing on the ambiguity inherent in data, rather than relying on probability or fuzzy logic. FST and RST represent two complementary approaches to addressing vagueness. While fuzzy sets (FzSs) focus on the continuity of membership, rough sets (RSs) emphasize the discreteness of knowledge granularity. Together, these theories provide a comprehensive framework for managing vagueness and uncertainty in various domains. Numerous studies have investigated the relationship between RST and FST, exploring their connections and distinctions. Dubois and Prade [6] made significant contributions to the field by successfully merging FST and RST, defining two novel constructs: rough fuzzy sets (RFzSs) and fuzzy rough sets (FzRSs).

RST is founded on a pair consisting of a universal set (US) and an equivalence relation (ER), where ER represents an indiscernibility relation. This relation models imprecise knowledge about the US. The theory utilizes upper and lower approximation sets, which are subsets of the US. Some authors have explored rough algebraic structures, while others have generalized RST by replacing the US with algebraic structures, investigating roughness within these structures. Biswas and Nanda [1] formulated the idea of rough groups and rough subgroups. N. Kuroki [7] presented rough ideals (RIs) in semigroups (SGs), extending the notion of ideals, and examined their features. Q. M. Xiao [15] made significant contribution by introducing two novel concepts: rough prime ideals (RPIs) and rough fuzzy prime ideals (RFzPIs) in SGs. Davvaz [4] established a link between RSs and ring theory, introduced rough subrings (resp. ideals) as an extension of subrings (resp. ideals) in rings. Osman [11] established the notions of RPIs and RFzPIs in a ring. Chinnadurai [3] found a relationship between RSs, fuzzy subsets and near-rings (NRs) and proposed the idea of fuzzy ideal (FzI) of a NR.

Brown [2] launched the theory of a near algebra (NA), a novel algebraic structure characterized by two binary operations. Specifically, a NA is an algebraic system that admits a field as a right operator domain and satisfying all ring postulates, potentially excepting one distributive property. Brown's pioneering

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work explored various types of NAs and their structural properties. The notion of fuzzy near algebra (FzNA) on a fuzzy field (FzF) is introduced and some findings on FzNA were obtained in [19]. Also fuzzy ideal (FzI) of a NA is defined and investigated the properties of this notion in [19]. Rajani [17] formulated RIs in a NA.

This paper explores the interconnections between RSs, FzSs and a NA. The manuscript is streamlined as follows: Section 2 gives an overview of basic definitions. Section 3 examines the characteristics of RFzSNAs in NAs and investigates the homomorphism properties of a RFzSNA.

2. Preliminaries

This section provides a concise review of the fundamental definitions underlying NAs, RSs and FzSs, drawing from a range of sources in the relevant literature.

Definition 2.1 *[2] A (right) near algebra N over a field F is a linear space (LS) N over F on which a multiplication is specified so that*
(i) N is a SG with respect to multiplication,
(ii) right distributivity holds for multiplication over addition,
i.e. $(\xi_1+\xi_2)\xi_3 = \xi_1\xi_3 + \xi_2\xi_3$ for all $\xi_1, \xi_2, \xi_3 \in N$ and
(iii) $k(\xi_1\xi_2) = (k\xi_1)\xi_2$ for all $k \in F, \xi_1, \xi_2 \in N$.

Remark 2.1 *N, N' are NAs over a field F throughout this manuscript.*

Definition 2.2 *[2] A non-empty subset S of N over a field F is known to be a sub near algebra (SNA) of N if it is a NA over F, inducing the operations from N.*

Definition 2.3 *[18] A mapping $\zeta : N \to N'$ is called a NA homomorphism if for every $\xi_1, \xi_2 \in N$ and $k \in F$:*
(i) $\zeta(\xi_1+\xi_2) = \zeta(\xi_1) + \zeta(\xi_2)$,
(ii) $\zeta(k\xi_1) = k\zeta(\xi_1)$,
(iii) $\zeta(\xi_1\xi_2) = \zeta(\xi_1)\zeta(\xi_2)$.
A one–one homomorphism which is onto is called an isomorphism.

Definition 2.4 *[20] Let A be a non–empty set. A fuzzy subset (FzSb) μ of A is a mapping $\mu : A \to [0,1]$.*

Definition 2.5 *[19] Let A and A' be two non–empty sets and $\zeta : A \to A'$ be a function. Let μ be a FzSb of A. Then $\zeta(\mu)$, the image of μ under ζ is a FzSb of A' given by*

$$\zeta(\mu)(\xi_2) = \begin{cases} \sup\{\mu(\xi_1) : \zeta(\xi_1) = \xi_2\} & \text{if } \zeta^{-1}(\xi_2) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\zeta^{-1}(\mu)$, the pre-image of μ under ζ is a FzSb of A specified by
$(\zeta^{-1}(\mu))(\xi_1) = \mu(\zeta(\xi_1))$ for every ξ_1 in A.

Definition 2.6 *[19] A FzSb μ of a field F is said to be a fuzzy field (FzF) of F, if it meets the specifications given below for every $\xi_1, \xi_2 \in F$:*
(i) $\mu(\xi_1 - \xi_2) \geq \mu(\xi_1) \wedge \mu(\xi_2)$,
(ii) $\mu(\xi_1\xi_2^{-1}) \geq \mu(\xi_1) \wedge \mu(\xi_2)$ for $\xi_2 \neq \emptyset$.
A FzF μ of F is indicated by (μ, F).

Definition 2.7 *[19] Let (ϕ, F) be a FzF. A FzSb μ of N is known as a fuzzy sub near algebra (FzSNA) of N over the FzF (ϕ, F) if it fulfils the following requirements for every $\xi_1, \xi_2 \in N$ and $\lambda \in F$:*
(i) $\mu(\xi_1+\xi_2) \geq \mu(\xi_1) \wedge \mu(\xi_2)$,
(ii) $\mu(\lambda\xi_1) \geq \phi(\lambda) \wedge \mu(\xi_1)$,
(iii) $\mu(\xi_1\xi_2) \geq \mu(\xi_1) \wedge \mu(\xi_2)$,
(iv) $\phi(1) \geq \mu(\xi_1)$ where 1 is the unity of F.

Definition 2.8 *[17] Let θ be an ER on N. Then θ is designated as a full congruence relation (FCR) if $(\xi_1, \xi_2) \in \theta$ leads to $(\xi_1 + \xi_3, \xi_2 + \xi_3), (k\xi_1, k\xi_2), (\xi_1\xi_3, \xi_2\xi_3)$ and $(\xi_3\xi_1, \xi_3\xi_2) \in \theta$ for all $k \in F$ and $\xi_3 \in N$.*

Theorem 2.1 *[17] Let θ be a FCR on N. If $\xi_1, \xi_2 \in N$ and $k \in F$, then*
(i) $[\xi_1]_\theta + [\xi_2]_\theta = [\xi_1 + \xi_2]_\theta$,
(ii) $[k\xi_1]_\theta = k[\xi_1]_\theta$.

Definition 2.9 *[17] A FCR θ on N is termed as complete congruence relation (CCR) if $[\xi_1\xi_2]_\theta = [\xi_1]_\theta[\xi_2]_\theta$ for all $\xi_1, \xi_2 \in N$.*

3. Rough Fuzzy Sub Near Algebras

Definition 3.1 *Let θ be a FCR on N and μ, a FzSb of N. Then we specify the FzSs $LApr_\theta(\mu)$ and $UApr_\theta(\mu)$ as follows:*

$$LApr_\theta(\mu)(\xi) = \bigwedge_{\nu \in [\xi]_\theta} \mu(\nu);\ UApr_\theta(\mu)(\xi) = \bigvee_{\nu \in [\xi]_\theta} \mu(\nu).$$

The FzSs $LApr_\theta(\mu)$ and $UApr_\theta(\mu)$ are known, in order as the θ-lower and θ-upper approximations of the FzS μ. $Apr_\theta(\mu) = (LApr_\theta(\mu), UApr_\theta(\mu))$ is designated as a rough fuzzy set (RFzS) with respect to θ if $LApr_\theta(\mu) \neq UApr_\theta(\mu)$.

Definition 3.2 *A FzSb μ of N is said to be an upper rough fuzzy sub near algebra (URFzSNA) if $UApr_\theta(\mu)$ is a FzSNA of N over a FzF (ϕ, F), and a lower rough fuzzy sub near algebra (LRFzSNA) if $LApr_\theta(\mu)$ is a FzSNA of N over the FzF (ϕ, F).*
Let μ be a FzSb of N and $Apr_\theta(\mu) = (LApr_\theta(\mu), UApr_\theta(\mu))$, a RFzS. If $LApr_\theta(\mu)$ and $UApr_\theta(\mu)$ are FzSNAs of N over the FzF (ϕ, F), then $(LApr_\theta(\mu), UApr_\theta(\mu))$ is known as a rough fuzzy sub near algebra (RFzSNA).

Theorem 3.1 *Let θ be a CCR on N. If μ is a FzSNA of N over a FzF (ϕ, F), then $UApr_\theta(\mu)$ is a FzSNA of N over the FzF (ϕ, F).*

Proof: For $\xi_1, \xi_2 \in N$ and $\lambda \in F$ we have,

$$\begin{aligned}
(i)\ UApr_\theta(\mu)(\xi_1 + \xi_2) &= \bigvee_{\xi_3 \in [\xi_1 + \xi_2]_\theta} \mu(\xi_3) \\
&= \bigvee_{\xi_3 \in [\xi_1]_\theta + [\xi_2]_\theta} \mu(\xi_3) \\
&= \bigvee_{\xi_1' \in [\xi_1]_\theta, \xi_2' \in [\xi_2]_\theta} \mu(\xi_1' + \xi_2') \\
&\geq \bigvee_{\xi_1' \in [\xi_1]_\theta, \xi_2' \in [\xi_2]_\theta} (\mu(\xi_1') \wedge \mu(\xi_2')) \\
&= \bigvee_{\xi_1' \in [\xi_1]_\theta} \mu(\xi_1') \wedge \bigvee_{\xi_2' \in [\xi_2]_\theta} \mu(\xi_2') \\
&= UApr_\theta(\mu)(\xi_1) \wedge UApr_\theta(\mu)(\xi_2).
\end{aligned}$$

$$\begin{aligned}
(ii)\ UApr_\theta(\mu)(\xi_1\xi_2) &= \bigvee_{\xi_3\in[\xi_1\xi_2]_\theta} \mu(\xi_3)\\
&= \bigvee_{\xi_3\in[\xi_1]_\theta[\xi_2]_\theta} \mu(\xi_3)\\
&= \bigvee_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} \mu(\xi_1'\xi_2')\\
&\geq \bigvee_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} (\mu(\xi_1')\wedge\mu(\xi_2'))\\
&= \bigvee_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\wedge \bigvee_{\xi_2'\in[\xi_2]_\theta} \mu(\xi_2')\\
&= UApr_\theta(\mu)(\xi_1)\wedge UApr_\theta(\mu)(\xi_2).\\
(iii)\ UApr_\theta(\mu)(\lambda\xi_1) &= \bigvee_{\xi_3\in[\lambda\xi_1]_\theta} \mu(\xi_3)\\
&= \bigvee_{\xi_3\in\lambda[\xi_1]_\theta} \mu(\xi_3)\\
&= \bigvee_{\xi_1'\in[\xi_1]_\theta} \mu(\lambda\xi_1')\\
&\geq \bigvee_{\xi_1'\in[\xi_1]_\theta} (\phi(\lambda)\wedge\mu(\xi_1'))\\
&= \phi(\lambda)\wedge \bigvee_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\\
&= \phi(\lambda)\wedge UApr_\theta(\mu)(\xi_1)\\
(iv)\ UApr_\theta(\mu)(\xi_1) &= \bigvee_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\\
&\leq \bigvee_{\xi_1'\in[\xi_1]_\theta} \phi(1)\\
&= \phi(1).
\end{aligned}$$

Hence $\phi(1)\geq UApr_\theta(\mu)(\xi_1)$. Therefore $UApr_\theta(\mu)$ is a FzSNA of N over the FzF (ϕ,F). □

Theorem 3.2 *Let θ be a CCR on N. If μ is a FzSNA of N over a FzF (ϕ,F), then $LApr_\theta(\mu)$ is a FzSNA of N over the FzF (ϕ,F).*

Proof: For $\xi_1,\xi_2\in N$ and $\lambda\in F$ we have,

$$\begin{aligned}
(i)\ LApr_\theta(\mu)(\xi_1+\xi_2) &= \bigwedge_{\xi_3\in[\xi_1+\xi_2]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_3\in[\xi_1]_\theta+[\xi_2]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} \mu(\xi_1'+\xi_2')\\
&\geq \bigwedge_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} (\mu(\xi_1')\wedge\mu(\xi_2'))\\
&= \bigwedge_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\wedge \bigwedge_{\xi_2'\in[\xi_2]_\theta} \mu(\xi_2')\\
&= LApr_\theta(\mu)(\xi_1)\wedge LApr_\theta(\mu)(\xi_2).
\end{aligned}$$

$$\begin{aligned}
(ii)\ LApr_\theta(\mu)(\xi_1\xi_2) &= \bigwedge_{\xi_3\in[\xi_1\xi_2]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_3\in[\xi_1]_\theta[\xi_2]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} \mu(\xi_1'\xi_2')\\
&\geq \bigwedge_{\xi_1'\in[\xi_1]_\theta,\xi_2'\in[\xi_2]_\theta} (\mu(\xi_1')\wedge\mu(\xi_2'))\\
&= \bigwedge_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\wedge \bigwedge_{\xi_2'\in[\xi_2]_\theta} \mu(\xi_2')\\
&= LApr_\theta(\mu)(\xi_1)\wedge LApr_\theta(\mu)(\xi_2).
\end{aligned}$$

$$
\begin{aligned}
(iii)\ LApr_\theta(\mu)(\lambda\xi_1) &= \bigwedge_{\xi_3\in[\lambda\xi_1]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_3\in\lambda[\xi_1]_\theta} \mu(\xi_3)\\
&= \bigwedge_{\xi_1'\in[\xi_1]_\theta} \mu(\lambda\xi_1')\\
&\geq \bigwedge_{\xi_1'\in[\xi_1]_\theta} (\phi(\lambda)\wedge\mu(\xi_1'))\\
&= \phi(\lambda)\wedge \bigwedge_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\\
&= \phi(\lambda)\wedge LApr_\theta(\mu)(\xi_1).\\
(iv)\ LApr_\theta(\mu)(\xi_1) &= \bigwedge_{\xi_1'\in[\xi_1]_\theta} \mu(\xi_1')\\
&\leq \bigwedge_{\xi_1'\in[\xi_1]_\theta} \phi(1)\\
&= \phi(1).
\end{aligned}
$$

Hence $\phi(1)\geq LApr_\theta(\mu)(\xi_1)$. Therefore $LApr_\theta(\mu)$ is a FzSNA of N over the FzF (ϕ,F).

□

Corollary 3.1 *If μ is a FzSNA of N over a FzF (ϕ,F), then $(LApr_\theta(\mu), UApr_\theta(\mu))$ is a RFzSNA of N over the FzF (ϕ,F).*

Proof: Direct consequence of theorems 3.1 and 3.2. □

Example 3.1 *Consider the LS $\mathbb{Z}_2\times\mathbb{Z}_2$ over the field Z_2. Specify multiplication on $\mathbb{Z}_2\times\mathbb{Z}_2$ as $(\psi_1,\psi_2)(\psi_1',\psi_2')=(\psi_1,\psi_2)$ for all $(\psi_1,\psi_2),(\psi_1',\psi_2')\in\mathbb{Z}_2\times\mathbb{Z}_2$. Then $\mathbb{Z}_2\times\mathbb{Z}_2$ is a NA over the field Z_2. Define θ on $\mathbb{Z}_2\times\mathbb{Z}_2$ as $(\psi_1,\psi_2)\theta(\psi_1',\psi_2')$ if and only if $\psi_1+_2\psi_2=\psi_1'+_2\psi_2'$. Then θ is a FCR on $\mathbb{Z}_2\times\mathbb{Z}_2$ with the equivalence classes $C_1=\{(0,0),(1,1)\}$ and $C_2=\{(0,1),(1,0)\}$. Let $F=Z_2$ and ϕ be a FzSb of F defined by*

$$\phi(\psi)=\begin{cases}0.9 & \text{if } \psi=0,\\ 0.8 & \text{if } \psi=1.\end{cases}$$

It is evident that (ϕ,F) is a FzF.
Define $\mu:\mathbb{Z}_2\times\mathbb{Z}_2\to[0,1]$ as $\mu(0,0)=0.2,\ \mu(0,1)=0.4,\ \mu(1,0)=0.6,\ \mu(1,1)=0.8$. Then μ is a FzSb of $\mathbb{Z}_2\times\mathbb{Z}_2$. We have $LApr_\theta(\mu)(\psi_1,\psi_2)=\{0.2,0.4\}$ and $UApr_\theta(\mu)(\psi_1,\psi_2)=\{0.6,0.8\}$ for all $(\psi_1,\psi_2)\in\mathbb{Z}_2\times\mathbb{Z}_2$. Here $UApr_\theta(\mu):\mathbb{Z}_2\times\mathbb{Z}_2\to[0,1]$ is defined by

$$UApr_\theta(\mu)(\psi_1,\psi_2)=\begin{cases}0.8 & \text{when } (\psi_1,\psi_2)=(0,0) \text{ or } (1,1),\\ 0.6 & \text{when } (\psi_1,\psi_2)=(0,1) \text{ or } (1,0).\end{cases}$$

We observe that $UApr_\theta(\mu)$ is a FzSNA of $\mathbb{Z}_2\times\mathbb{Z}_2$ over the FzF (ϕ,F).

Definition 3.3 *Let μ be a FzSb of N. Then the sets*
$\mu_t=\{\xi_1\in N\mid\mu(\xi_1)\geq t\},\ \mu_t^s=\{\xi_1\in N\mid\mu(\xi_1)>t\}$ where $t\in[0,1]$
are specifically designated in order as, t- lower level subset and t- strong lower level subset of μ.

Theorem 3.3 *Let θ be a FCR on N. If μ is a FzSb of N and $t\in[0,1]$, then*
(i) $(LApr_\theta(\mu))_t=LApr_\theta(\mu_t)$,
(ii) $(UApr_\theta(\mu))_t^s=UApr_\theta(\mu_t^s)$.

Proof: (i) We have,

$$
\begin{aligned}
\xi_1\in(LApr_\theta(\mu))_t &\iff LApr_\theta(\mu)(\xi_1)\geq t\\
&\iff \bigwedge_{\xi_1'\in[\xi_1]_\theta}\mu(\xi_1')\geq t\\
&\iff \text{for all } \xi_1'\in[\xi_1]_\theta,\ \mu(\xi_1')\geq t\\
&\iff [\xi_1]_\theta\subseteq\mu_t\\
&\iff \xi_1\in UApr_\theta(\mu_t).
\end{aligned}
$$

(ii) We have,

$$\begin{aligned}
\xi_1 \in (UApr_\theta(\mu))^s_t &\iff UApr_\theta(\mu)(\xi_1) > t \\
&\iff \bigvee_{\xi_1' \in [\xi_1]_\theta} \mu(\xi_1') > t \\
&\iff \text{there exists } \xi_1' \in [\xi_1]_\theta \text{ such that } \mu(\xi_1') > t \\
&\iff [\xi_1]_\theta \cap \mu^s_t \neq \emptyset \\
&\iff \xi_1 \in UApr_\theta(\mu^s_t).
\end{aligned}$$

□

Remark 3.1 *From here on, θ and ρ indicate FCRs on N and N' respectively.*

Theorem 3.4 *Let $\zeta : N \to N'$ be an onto NA homomorphism and μ be a FzSb of N. Then*
(i) $\zeta(UApr_\theta(\mu)) = UApr_\rho(\zeta(\mu))$;
(ii) $\zeta(LApr_\theta(\mu)) \subseteq LApr_\rho(\zeta(\mu))$. If ζ is one-one, then $\zeta(UApr_\theta(\mu)) = UApr_\rho(\zeta(\mu))$.

Proof: (i) For $\xi_1 \in N'$, we have

$$\begin{aligned}
\zeta(UApr_\theta(\mu))(\xi_1) &= \bigvee_{\zeta(\xi_1')=\xi_1} UApr_\theta(\mu)(\xi_1') \\
&= \bigvee_{\zeta(\xi_1')=\xi_1} \bigvee_{\xi_3 \in [\xi_1']_\theta} \mu(\xi_3) \\
&= \bigvee_{\zeta(\xi_1')=\xi_1} \bigvee_{\xi_1' \in [\xi_3]_\theta} \mu(\xi_1') \\
&= \bigvee_{\xi_1' \in [\xi_3]_\theta} \bigvee_{\zeta(\xi_1')=\xi_1} \mu(\xi_1') \\
&= \bigvee_{\zeta(\xi_1') \in [\zeta(\xi_3)]_\rho} \zeta(\mu)(\xi_1) \\
&= \bigvee_{\xi_1 \in [\zeta(\xi_3)]_\rho} \zeta(\mu)(\xi_1) \\
&= \bigvee_{\zeta(\xi_3) \in [\xi_1]_\rho} \zeta(\mu)(\zeta(\xi_3)) \\
&= UApr_\rho(\zeta(\mu))(\xi_1).
\end{aligned}$$

(ii) The proof is analogous to (i). □

Theorem 3.5 *Let μ be a FzSb of N. Then μ is a FzSNA of N over a FzF (ϕ, F) if and only if μ_t and μ^s_t are, if they are non–empty, SNAs of N over the field $\phi_t = \{\xi_1 \in F \mid \phi(\xi_1) \geq t, t \in [0,1]\}$ for every $t \in [0,1]$.*

Proof: Presume that μ is a FzSNA of N over the FzF (ϕ, F). We have,
$\mu_t = \{\xi_1 \in N \mid \mu(\xi_1) \geq t\}$ and $\phi_t = \{\xi_1 \in F \mid \phi(\xi_1) \geq t, t \in [0,1]\}$.
Let $\xi_1, \xi_2 \in \mu_t$ and $\lambda \in \phi_t$. Then $\mu(\xi_1), \mu(\xi_2) \geq t$. Now,
(i) $\mu(\xi_1 + \xi_2) \geq \mu(\xi_1) \wedge \mu(\xi_2) \leq t$. So $\xi_1 + \xi_2 \in \mu_t$.
(ii) $\mu(\xi_1\xi_2) \geq \mu(\xi_1) \wedge \mu(\xi_2) \leq t$. So $\xi_1\xi_2 \in \mu_t$.
(iii) $\mu(\lambda\xi_1) \geq \phi(\lambda) \wedge \mu(\xi_1) \leq t$. So $\lambda\xi_1 \in \mu_t$.
Therefore μ_t is a SNA of N over the field ϕ_t.
On the other hand, assume that μ_t is a SNA of N over the field ϕ_t.
Presume that, $\mu(\xi_1 + \xi_2) < (\mu(\xi_1) \wedge \mu(\xi_2))$. Let $u = \frac{1}{2}(\mu(\xi_1 + \xi_2) + (\mu(\xi_1) \wedge \mu(\xi_2)))$. Then $\mu(\xi_1 + \xi_2) < u < (\mu(\xi_1) \wedge \mu(\xi_2))$. This implies $\mu(\xi_1 + \xi_2) < u$ and $u < (\mu(\xi_1) \wedge \mu(\xi_2))$ and hence $\xi_1 + \xi_2 \notin \mu_u$. Also $\mu(\xi_1) > u$ and $\mu(\xi_2) > u$, this implies $\xi_1, \xi_2 \in \mu_u$, which yields $\xi_1 + \xi_2 \in \mu_u$. Which is absurd and hence $\mu(\xi_1 + \xi_2) \geq (\mu(\xi_1) \wedge \mu(\xi_2))$. — (1)
Assume that, $\mu(\xi_1\xi_2) < (\mu(\xi_1) \wedge \mu(\xi_2))$. Let $v = \frac{1}{2}(\mu(\xi_1\xi_2) + (\mu(\xi_1) \wedge \mu(\xi_2)))$. Then $\mu(\xi_1\xi_2) < v < (\mu(\xi_1) \wedge \mu(\xi_2))$. This implies $\mu(\xi_1\xi_2) < v$ and $v < (\mu(\xi_1) \wedge \mu(\xi_2))$ and hence $\xi_1\xi_2 \notin \mu_v$. Also $\mu(\xi_1) > v$ and $\mu(\xi_2) > v$, this implies $\xi_1, \xi_2 \in \mu_v$, which yields $\xi_1\xi_2 \in \mu_v$. Which is a fallacy and hence $\mu(\xi_1 + \xi_2) \geq$

$(\mu(\xi_1) \wedge \mu(\xi_2))$. — (2)
Let $\lambda \in F$. Assume that, $\mu(\lambda\xi_1) < (\phi(\lambda) \wedge \mu(\xi_1))$. Let $w = \frac{1}{2}(\mu(\lambda\xi_1) + (\phi(\lambda) \wedge \mu(\xi_1)))$. Then, $\mu(\lambda\xi_1) < w < (\phi(\lambda) \wedge \mu(\xi_1))$. This implies $\mu(\lambda\xi_1) < w$ and $w < (\phi(\lambda) \wedge \mu(\xi_1))$ and hence $\lambda\xi_1 \notin \mu_w$. Also $\phi(\lambda) > w$ and $\mu(\xi_1) > w$, this implies $\lambda \in \phi_w$ and $\xi_1 \in \mu_w$. Which yields $\lambda\xi_1 \in \mu_w$ which is illogical and hence $\mu(\lambda\xi_1) \geq (\phi(\lambda) \wedge \mu(\xi_1))$. — (3)
Assume $\phi(1) < \mu(\xi_1)$. Let $s = \frac{1}{2}(\phi(1) + \mu(\xi_1))$. Then $\phi(1) < s$ and $s < \mu(\xi_1)$. Therefore $1 \notin \phi_s$ and $\xi_1 \in \mu_s$ which is a contradiction to the fact that μ_s is a SNA of N over ϕ_s and hence $\phi(1) \geq \mu(\xi_1)$. —(4)
From (1), (2), (3), (4), μ is a FzSNA over the FzF (ϕ, F). Similarly we can show that μ is a FzSNA over the FzF (ϕ, F) if and only if μ_t^s is a SNA of N over the field ϕ_t. □

Theorem 3.6 *Let $\zeta : N \to N'$ be a NA isomorphism and μ be a FzSb of N. Then*
(i) $UApr_\theta(\mu)$ is a FzSNA of N over a FzF (ϕ, F) if and only if $UApr_\rho(\zeta(\mu))$ is a FzSNA of N' over the FzF (ϕ, F).
(ii) $LApr_\theta(\mu)$ is a FzSNA of N over a FzF (ϕ, F) if and only if $LApr_\rho(\zeta(\mu))$ is a FzSNA of N' over the FzF (ϕ, F).

Proof: (i) $UApr_\theta(\mu)$ is a FzSNA of N over a FzF (ϕ, F) if and only if $(UApr_\theta(\mu))_t^s$ is, if it non-empty, a SNA of N over the field ϕ_t for every $t \in [0, 1]$.
We have, $(UApr_\theta(\mu))_t^s$=$UApr_\theta(\mu_t^s)$ and $UApr_\theta(\mu_t^s)$ is a SNA of N if and only if $\zeta(UApr_\theta(\mu_t^s))$ is a SNA of N'.
Also, $\zeta(UApr_\theta(\mu_t^s)) = UApr_\rho(\zeta(\mu_t^s)) = UApr_\rho(\zeta(\mu))_t^s = (UApr_\rho(\zeta(\mu)))_t^s$.
Thus $(UApr_\rho(\zeta(\mu)))_t^s$ is a SNA of N' for every $t \in [0, 1]$ if and only if $UApr_\rho(\zeta(\mu))$ is a FzSNA of N' over the FzF (ϕ, F).
(ii) The proof is analogous to (i). □

Theorem 3.7 *Let $\zeta : N \to N'$ be an onto homomorphism. If σ is an URFzSNA of N' over a FzF (ϕ, F), then $\zeta^{-1}(\sigma)$ is a URFzSNA of N over the FzF (ϕ, F).*

Proof: Let σ be an URFzSNA of N' over the FzF (ϕ, F). Then $UApr_\rho(\sigma)$ is a FzSNA of N' over the FzF (ϕ, F). We now show that $UApr_\theta(\zeta^{-1}(\sigma))$ is a FzSNA of N over the FzF (ϕ, F). We have $\zeta^{-1}(UApr_\rho(\sigma))$=$UApr_\theta(\zeta^{-1}(\sigma))$. Let $\xi_1, \xi_2 \in N$ and $\lambda \in F$.Then,

$$\begin{aligned}
(i)\ \zeta^{-1}(UApr_\rho(\sigma))(\xi_1 + \xi_2) &= UApr_\rho(\sigma)(\zeta(\xi_1 + \xi_2)) \\
&= UApr_\rho(\sigma)(\zeta(\xi_1) + \zeta(\xi_2)) \\
&\geq UApr_\rho(\sigma)(\zeta(\xi_1)) \wedge UApr_\rho(\sigma)(\zeta(\xi_2)) \\
&= \zeta^{-1}(UApr_\rho(\sigma))(\xi_1) \wedge \zeta^{-1}(UApr_\rho(\sigma))(\xi_2). \\
(ii)\ \zeta^{-1}(UApr_\rho(\sigma))(\xi_1\xi_2) &= UApr_\rho(\sigma)(\zeta(\xi_1\xi_2)) \\
&= UApr_\rho(\sigma)(\zeta(\xi_1)\zeta(\xi_2)) \\
&\geq UApr_\rho(\sigma)(\zeta(\xi_1)) \wedge UApr_\rho(\sigma)(\zeta(\xi_2)) \\
&= \zeta^{-1}(UApr_\rho(\sigma))(\xi_1) \wedge \zeta^{-1}(UApr_\rho(\sigma))(\xi_2). \\
(iii)\ \zeta^{-1}(UApr_\rho(\sigma))(\lambda\xi_1) &= UApr_\rho(\sigma)(\zeta(\lambda\xi_1)) \\
&= UApr_\rho(\sigma)(\lambda\zeta(\xi_1)) \\
&\geq \phi(\lambda) \wedge UApr_\rho(\sigma)(\zeta(\xi_1)) \\
&= \phi(\lambda) \wedge \zeta^{-1}(UApr_\rho(\sigma))(\xi_1).
\end{aligned}$$

$UApr_\rho(\sigma)$ is a FzSNA of N' over the FzF (ϕ, F). For all $\xi_3 \in N'$, we have $\phi(1) \geq UApr_\rho(\sigma)(\xi_3)$. Then for every $\xi_1 \in N, \phi(1) \geq UApr_\rho(\sigma)(\zeta(\xi_1)) = \zeta^{-1}(UApr_\rho(\sigma))(\xi_1)$. Hence $\zeta^{-1}(UApr_\rho(\sigma))$=$UApr_\theta(\zeta^{-1}(\sigma))$ is a FzSNA of N over the FzF (ϕ, F). □

Theorem 3.8 *Let $\zeta : N \to N'$ be a NA isomorphism. If σ is a LRFzSNA of N' over a FzF (ϕ, F), then $\zeta^{-1}(\sigma)$ is a LRFzSNA of N over the FzF (ϕ, F).*

Theorem 3.9 *Let $\zeta : N \to N'$ be an onto NA isomorphism. If σ is a URFzSNA of N over a FzF (ϕ, F), then $\zeta(\sigma)$ is an URFzSNA of N' over the FzF (ϕ, F).*

Proof: σ is an URFzSNA of N over the FzF (ϕ, F). Then $UApr_\theta(\sigma)$ is a FzSNA of N over the FzF (ϕ, F). We now show that $\zeta(\sigma)$ is an URFzSNA of N' over the FzF (ϕ, F). We have $UApr_\rho(\zeta(\sigma)) = \zeta(UApr_\theta(\sigma))$. Hence it is enough to show that $\zeta(UApr_\theta(\sigma))$ is a FzSNA of N'. Let $\xi_1', \xi_2' \in N'$ and $\lambda \in F$. Then there exist $\xi_1, \xi_2 \in N$ such that $\xi_1' = \zeta(\xi_1)$, $\xi_2' = \zeta(\xi_2)$.

$$\begin{aligned}
(i)\ \zeta(UApr_\theta(\sigma))(\xi_1' + \xi_2') &= \sup\{UApr_\theta(\sigma)(\xi_3) : \xi_3 \in \zeta^{-1}(\xi_1' + \xi_2')\} \\
&\geq \sup\{UApr_\theta(\sigma)(\xi_1 + \xi_2) : \zeta(\xi_1) = \xi_1',\ \zeta(\xi_2) = \xi_2'\} \\
&\geq \sup\{UApr_\theta(\sigma)(\xi_1) \wedge UApr_\theta(\sigma)(\xi_2)\} \\
&= \sup\{UApr_\theta(\sigma)(\xi_1) : \zeta(\xi_1) = \xi_1'\} \\
&\quad \wedge \sup\{UApr_\theta(\sigma)(\xi_2) : \zeta(\xi_2) = \xi_2'\} \\
&= \zeta(UApr_\theta(\sigma))(\xi_1') \wedge \zeta(UApr_\theta(\sigma))(\xi_2')
\end{aligned}$$

$$\begin{aligned}
(ii)\ \zeta(UApr_\theta(\sigma))(\xi_1'\xi_2') &= \sup\{UApr_\theta(\sigma)(\xi_3) : \xi_3 \in \zeta^{-1}(\xi_1'\xi_2')\} \\
&\geq \sup\{UApr_\theta(\sigma)(\xi_1\xi_2) : \zeta(\xi_1) = \xi_1',\ \zeta(\xi_2) = \xi_2'\} \\
&\geq \sup\{UApr_\theta(\sigma)(\xi_1) \wedge UApr_\theta(\sigma)(\xi_2)\} \\
&= \zeta(UApr_\theta(\sigma))(\xi_1') \wedge \zeta(UApr_\theta(\sigma))(\xi_2')
\end{aligned}$$

$$\begin{aligned}
(iii)\ \zeta(UApr_\theta(\sigma))(\lambda\xi_1') &= \sup\{UApr_\theta(\sigma)(\xi_3) : \xi_3 \in \zeta^{-1}(\lambda\xi_1')\} \\
&\geq \sup\{UApr_\theta(\sigma)(\lambda\xi_3) : \zeta(\xi_3) = \xi_1'\} \\
&\geq \sup\{\phi(\lambda) \wedge UApr_\theta(\sigma)(\xi_3) : \zeta(\xi_3) = \xi_1'\} \\
&= \phi(\lambda) \wedge \sup\{UApr_\theta(\sigma)(\xi_3) : \zeta(\xi_3) = \xi_1'\} \\
&= \phi(\lambda) \wedge \zeta(UApr_\theta(\sigma))(\xi_1')
\end{aligned}$$

Also, $\phi(1) \geq UApr_\theta(\sigma)(\xi_1)$ for all $\xi_1 \in N$. Hence for all $\xi_1' \in N'$, $\phi(1) \geq \zeta(UApr_\theta(\sigma))(\xi_1')$. Therefore, $\zeta(UApr_\theta(\sigma))$ is a FzSNA of N', and hence $UApr_\rho(\zeta(\sigma))$ is a FzSNA of N' over (ϕ, F). □

Theorem 3.10 *Let $\zeta : N \to N'$ be a NA isomorphism. If σ is a LRFzSNA of N over a FzF (ϕ, F), then $\zeta(\sigma)$ is a LRFzSNA of N' over the FzF (ϕ, F).*

4. Conclusion

This manuscript presented a comprehensive examination of RFzSNAs in the context of NAs. We systematically investigated the distinctive characteristics of these algebraic structures, provided illustrative examples to support our findings. Additionally, we explored the connections between U(L)RFzSNAs and their homomorphism images.

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