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Numerical Simulation of Time-Dependent Parabolic Differential-Difference Equations cith Singular Perturbation Using an Adaptive Spline

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ABSTRACT: The work introduces a numerical scheme tailored for time-dependent parabolic singularly perturbed differential–difference equations (SPPDDEs), specifically those containing small delay terms in both the convection and diffusion components. When the delay or advance parameters are much smaller than the perturbation parameter, the delay terms are approximated using a Taylor series expansion. The approach applies the backward Euler method for time discretization and employs an adaptive spline technique on a uniform spatial grid. This combination yields first-order accuracy in time and second-order accuracy in space. A series of numerical experiments is carried out to support the theoretical analysis and to benchmark the method against several established techniques. The results indicate that the proposed scheme offers enhanced precision and improved convergence relative to existing methods.

Keywords: Adaptive spline, parabolic differential difference equation, fitting factor, truncation error.

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1. Introduction

The researchers have studied two first order upwind and modified upwind in [22] for SPPDDE. The pseudo-parabolic singularly perturbed equation with initial jump is given parameter uniformly convergent numerical techniques in [1-2]. The methods presented by the authors in [3-5] incorporated shift terms of arbitrary order for a problem of similar nature. Similarly, the SPPDDEs with interior layer and a significant response term delay was also considered by the authors in [6]. In [11], authors have taken the same problem and provided an approach based on the fitted mesh and B−spline collocation approch that increased the rate of convergence. A three step Taylor−Galerkin FEM has been developed and is discussed in [12] with regards to stability and uniform convergence, A series of investigations were conducted on creating an approximation of the SPPDDE solution by Lange and Miura [13–16]. Most of the prior research [18,19,20] focused on the existence of uniqueness of similar problems before Lange and Miura.

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2. Description of the Problem

Consider a class of SPPDDE of the form

$$\frac{\partial Y}{\partial t} - \varepsilon^2 \frac{\partial^2 Y}{\partial r^2} + \mathcal{H}(r) \frac{\partial Y}{\partial r} + \mathcal{I}(r)\ Y(r-\delta, t) + \mathcal{J}(r)\ Y(r,t) \tag{2.1}$$
$$+ \mathcal{L}(r)\ Y(r+\eta, t) = g(r,t)$$

Here $(r,t) \in \Phi = \Upsilon \times \Omega, \Upsilon = (0,1),\ \Omega = (0,T]$ with

$$Y(r,0) = Y_0(r),\ \dagger \in \bar{\Upsilon}$$

$$Y(r,t) = \phi(v,t),\ (r,t) \in \Phi_L = [-\delta, 0] \times (0,1]$$
$$Y(r,t) = \varphi(r,t),\ (r,t) \in \Phi_R = [1, 1+\eta] \times (0,1] \tag{2.2}$$

and $\varepsilon \in (0,1]$and $\delta, \eta = o(\varepsilon)$.
The functions $\mathcal{H}(r), \mathcal{I}(r), \mathcal{J}(r), \mathcal{L}(r), g(r,t), \omega_0,\ \phi$ and φ are assumed to be smooth and not influenced by ε. Also, assume the coefficient of reaction term satisfied by

$$\mathcal{I}(r) + \mathcal{J}(r) + \mathcal{L}(r) \geq \gamma > 0, \forall\ r \in \bar{\Upsilon}$$

for some constant γ, to ensure an oscillation-free solution. When $\delta = 0, \eta = 0$, Eq. (2.1) reduces to parabolic perturbation problem involving one small parameter ε and layers appears based on the sign of the convection coefficient term, i.e., the layer appears on the right side of the domain Φ_R if $\mathcal{H}(r) \geq 0$, on the domain's left side Φ_L if $\mathcal{H}(r) \leq 0$ and if $\mathcal{H}(r)$ changes sign in Φ , an interior layer may appear [1]. When shift parameters η and δ are small order of ε, the term containing retarded terms can be expanded using Taylor's series expansion [6], so we approximate

$$Y(r-\delta, t) = Y(r,t) - \delta\ Y_r(r,t) + \frac{\delta^2}{2!} Y_{rr}(r,t) - O(\delta^3) \tag{2.3}$$

$$Y(r+\eta, t) = Y(r,t) + \eta\ Y_r(r,t) + \frac{\eta^2}{2!} Y_{rr}(r,t) + O(\eta^3) \tag{2.4}$$

Substituting (2.3) and (2.4) in Eq.(2.1) we obtain

$$\frac{\partial Y}{\partial t} - \varepsilon^2 C_\varepsilon \frac{\partial^2 Y}{\partial r^2} + \alpha(r) \frac{\partial Y}{\partial r} + \beta(r)\ Y(r,t) = g(r,t) \tag{2.5}$$

subject to the conditions

$$Y(r,0) = Y_0(v),\ r \in \bar{\Upsilon},$$
$$Y(0,t) = \phi(0,t),\ t \in (0,T]$$
$$Y(1,t) = \varphi(1,t),\ t \in (0,T] \tag{2.6}$$

where $C_\varepsilon = \left(1 + \frac{\delta^2}{2\varepsilon^2}\mathcal{I} + \frac{\eta^2}{2\varepsilon^2}\mathcal{L}\right), \alpha(r) = \frac{\mathcal{H}(r) - \delta\ \mathcal{I}(r) + \eta\ \mathcal{L}(r)}{C_\varepsilon}$ and $\beta(r) = \frac{\mathcal{I}(r)+\mathcal{J}(r)+\mathcal{L}(r)}{C_\varepsilon}$ Since $\delta, \eta = o(\varepsilon)$, Eq's. (2.1) and (2.5) are asymptotically equal, and the distinction between the equations is $O(\delta^3, \eta^3)$. we assume that $0 < C_\varepsilon(r) < \varepsilon^2 - \delta^2 P_1 - \eta^2 P_2 = C_\varepsilon$ where $\mathcal{I}(r) \geq 2P_1$ and $\mathcal{L}(r) \geq 2P_2$ for P_1 and P_2 are constants. It is also assumed that $\alpha(r) \geq \alpha^* > 0$ on Υ for some constant. As a result, the boundary layer is positioned on the right-hand side of the rectangle domain Φ. If $\alpha(r) \leq \alpha^* < 0$ on Υ the boundary layer show at the left side of the domain Φ. Furthermore, we assumed that $\beta(r) \geq \beta^* > 0$ for all $r \in \bar{\Upsilon}$. We also include the compatibility conditions are

$$\omega_0(0) = \phi(0,0) \text{ and } \omega_0(1) = \varphi(1,0) \tag{2.7}$$

and

$$\frac{\partial \phi(0,0)}{\partial\ t} - C_\varepsilon \frac{\partial^2 \omega_0(0,0)}{\partial r^2} + \mathcal{H}(0) \frac{\partial \omega_0(0,0)}{\partial r} + \mathcal{I}(0)\phi(-\delta, 0)$$

$$+\mathcal{J}(0)\,\omega_0(0,0)+\mathcal{L}(0)\ \omega_0(\eta,0)=F(0,0),$$

$$\frac{\partial\varphi(1,0)}{\partial\, t}-C_\varepsilon\frac{\partial^2\omega_0(1,0)}{\partial r^2}+\mathcal{H}(1)\frac{\partial\omega_0(1,0)}{\partial r}+\mathcal{I}(1)\,\omega_0(1-\delta,0)$$
$$+\mathcal{J}(1)\,\omega_0(1,0)+\mathcal{L}(1)\,\varphi(1+\eta,0)=F(1,0) \tag{2.8}$$

so that the data is consistent at both ends $(0,0)$ and $(1,0)$. Note that we assume $\phi(r,t),\varphi(r,t)$ and $\omega_0(r)$ are regular in order to satisfy Eq. (2.8). When the layer appears on the rectangular domain's right side, we can state that, assuming the compatibility conditions in Eq's.(2.7) and (2.8), a constant M, independent of ε, exists for all $(r,t)\in\bar{\Phi}$ such that

$$|Y(r,t)-Y(r,0)|=|Y(v,t)-\omega_0(v)|\le\ Mt \tag{2.9}$$

$$|Y(r,t)-Y(0,t)|=|Y(r,t)-\phi(0,t)|\le\ M(1-r) \tag{2.10}$$

One can see [7], for the Eq. (2.9) and Eq. (2.10).

3. Properties of Continuous Problem

Lemma 3.1 *The bound on the solution $Y(r,t)$ of the problem (2.5) – (2.6) is given by*

$$|Y(r,t)|\le\ M,\ \ (r,t)\in\bar{\Phi}$$

Proof: Refer [9].

Lemma 3.2 *Let $Y(r,t)\in\ C^{2,1}(\Omega)$. If $Y(r,t)\ge 0\ \forall\ (r,t)\in\partial\Omega$ then $L_{\varepsilon,\Delta,\eta}Y(v,t)\ge 0\ \ \forall\ (v,t)\in\Omega$ implies that $Y(r,t)\ge 0\ \forall\ (r,t)\in\Phi$.*
Proof: Refer [9].

4. Discretization of Time Variable

To discretize the problem (2.5) - (2.6) in the direction of time variable, partitioned the time domain $\bar{\Phi}_t$ in to M equal parts with uniform step size Δt on $\bar{\Phi}_t^M=[(r,t_j):v\in\Upsilon,t_j=j\Delta t=j(\frac{T}{M}),\forall\ 0\le j\le M]$. Now using back Euler method, we have a system of differential equations

$$\frac{Y^j(r)-Y^{j-1}(r)}{\Delta t}-\varepsilon^2C_\varepsilon\ Y^j_{rr}(r)+\alpha(r)\ \ Y^j_r(r)+\beta(r)\ \ Y^j(r)=g(r,t_j)$$

By simplifying, we attain a system of discrete linear ordinary differential equations

$$-\varepsilon^2C_\varepsilon\ Y^j_{rr}(r)+\alpha(r)\ \ Y^j_r(r)+\widetilde{\beta}(r)\ \ Y^j(r)=G(r,t_j) \tag{4.1}$$

with

$$Y(r,0)=\omega_0(r),r\in\bar{\Upsilon},Y(0,t_j)=\phi(0,t_j),\ \ \forall\ 0\le\ j\le\ M$$
$$Y(1,t_j)=\psi(1,t_j),\ \ \forall\ 0\le\ j\le\ M \tag{4.2}$$

where

$$\widetilde{\beta}(r)=(\beta(r)+\frac{1}{\Delta t}),\ G(r,t_j)=g(r,t_j)+\frac{Y^{j-1}(r)}{\Delta t}$$

The local truncation error of the problem Eq.(4.1) in temporal discretization is defined as $e_j=Y(r,t_j)-\hat{Y}(r,t_j)$, where $\hat{Y}(r,t_j)$ is the solution of the problem

$$\mathcal{L}_{\theta_\varepsilon,\Delta,\eta}\hat{Y}(r)=-C_\varepsilon\frac{\partial^2\hat{Y}}{\partial r^2}+\alpha(r)\ \ \frac{\partial\hat{Y}}{\partial r}+\widetilde{\beta}(v)\ \ \hat{Y}=G(r,t_j),r\in\Upsilon,\ \ 0\le\ j\le\ M, \tag{4.3}$$

with

$$\hat{Y}(0)=\phi(0,t_j),\ \hat{Y}(1)=\psi(0,t_j),\ 0\le\ j\le\ M \tag{4.4}$$

Lemma 4.1 *The estimated local error in the time is given by*

$$e_j \leq K(\Delta t)^2 \tag{4.5}$$

where $e_j = Y(r,t_j) - \hat{Y}(r,t_j)$ *is the local error estimate in the time at* j^{th} *time level.*

Lemma 4.2 *The estimated global error in the time direction is given by Here,* K *is a positive constant that does not depend on* ε *or* Δt.
Proof: One can refer [9].

Lemma 4.3 *The exact solution* $\hat{Y}(r,t_j)$ *of semi discretization Eq.(4.3), and its derivatives satisfies*

$$\left|\frac{d^i\hat{Y}(r,t_j)}{dr^i}\right| \leq \ K\left[1 + C_\varepsilon^{-i} exp\left(-\alpha^*\left(\frac{1-v}{C_\varepsilon}\right)\right)\right], r \in \Upsilon \text{ for } \ 0 \leq i \leq 3 \tag{4.6}$$

Proof: [22] provides a detailed proof of this lemma.

5. Discretization of Spatial Variable with an Adaptive Cubic Spline Scheme

In this section, we proposed an adaptive spline-based numerical algorithm for spatial discretization of differential equations resultant from time discretization with uniform mesh,

$$-C_\varepsilon\ Y^{''}(r) + \alpha(v)\ Y^{'}(r) + \widetilde{\beta}(r)\ Y(r) = G(r), \quad v \in \Upsilon \tag{5.1}$$

with

$$Y(0) = \phi(0), \qquad Y(1) = \psi(1). \tag{5.2}$$

Now, consider a uniform mesh with pivotal points r_i in the interval $[0,1]$ such that $\Upsilon : 0 = r_0 < r_1 < r_2 \ldots < r_{L-1} < r_L = 1$, where $v_i = ih,\ h = \frac{1}{L}\ \forall\ i = 0,1,2,3,\ldots,L$. A function $Q_\Upsilon(r,\tau)$ of class $C^2[0,1]$, which interpolates $Y(r)$ at the grid points r_i depending on a parameter τ, reduces to a cubic spline $Q_\Upsilon(r)$, in [0, 1] as $\tau \to 0$, is referred to as an adaptive spline. If $Q_\Upsilon(r,\tau)$ is an adaptive spline function, then according to [14], we have

$$C_\varepsilon\ Q_\Upsilon^{''}(r,\tau) - \alpha\ Q_\Upsilon^{'}(r,\tau) = \frac{r - r_{i-1}}{h}(C_\varepsilon\ M_i - \alpha\ m_i) + \tag{5.3}$$
$$\frac{r_i - r}{h}(C_\varepsilon\ M_{i-1} - \alpha\ m_{i-1})$$

where $r_{i-1} \leq r \leq r_i$,, C_ε is constant and $Q_\Upsilon^{'}(r,\tau) = m_i,\ \ Q_\Upsilon^{''}(r,\tau) = M_i$. Solving Eq.(5.3) and using the conditions of interpolation i.e.,$Q_\Upsilon(r_{i-1},\tau) = Y_{i-1}, Q_\Upsilon(r_i,\tau) = Y_i$, we have

$$Q_\Upsilon(r,\tau) = A_i + B_i e^{\tau\ z} - \frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2 z^2 + \tau\ z + 1\right]\left(M_i - \frac{\tau}{h}m_i\right)$$
$$+\frac{h^2}{\tau^3}\left[\frac{1}{2}\tau^2(1-z)^2 + \tau\ (1- z) + 1\right]\left(M_{i-1} - \frac{\tau}{h}m_{i-1}\right) \tag{5.4}$$

where

$$A_i\,(e^\tau - 1) = -Y_i + Y_{i-1}e^\tau - \frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} + \tau + 1\right) - \tau\ e^\tau\right]\left(M_i - \frac{\tau}{h}m_i\right)$$
$$-\frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} - \tau + 1\right) - \tau\right]\left(M_{i-1} - \frac{\tau}{h}m_{i-1}\right)$$
$$B_i\,(e^\tau - 1) = Y_i - Y_{i-1} + \frac{h^2}{\tau^2}\left[\left(\frac{\tau}{2} + 1\right)\right]\left(M_i - \frac{\tau}{h}m_i\right) +$$
$$\left[\left(\frac{\tau}{2} - 1\right)\right]\left(M_{i-1} - \frac{\tau}{h}m_{i-1}\right)$$

$\tau=\frac{\alpha\ h}{C_\varepsilon}$ and $z=\frac{r-r_{i-1}}{h}$. The spline function $Q_\Upsilon(r,\tau)$ on the interval $[r_i,r_{i+1}]$ is achieved by replacing i by $(i+1)$ in Eq.(5.4) and by using the first or second derivative continuity condition of $Q_\Upsilon(r,\tau)$at $r=r_i$, we get the relationship below:

$$\begin{aligned}&\left(M_{i+1}-\frac{\tau}{h}m_{i+1}\right)\left[e^{-\tau}\left(\frac{\tau^2}{2}+\tau+1\right)-1\right]+\left(M_i-\frac{\tau}{h}m_i\right)\\&\{e^{-\tau}\left(\frac{\tau^2}{2}-\tau-2\right)+\left(-\frac{\tau^2}{2}-\tau+2\right)\}+\left(M_{i-1}-\frac{\tau}{h}m_{i-1}\right)\\&\left[e^{-\tau}-1+\tau-\frac{\tau^2}{2}\right]=-\frac{\tau^2}{h^3}\left[e^{-\tau}Y_{i+1}-\left(1+e^{-\tau}\right)Y_i+Y_{i-1}\right]\end{aligned}\tag{5.5}$$

Some additional relations for the proposed splines are listed below

$$(i)\ m_{i-1}=-h(\tilde{P}_1M_{i-1}+\tilde{P}_2M_i)+\frac{(Y_i-Y_{i-1})}{h}$$

$$(ii)\ m_i=h(\tilde{P}_3M_{i-1}+\tilde{P}_4M_i)+\frac{(Y_i-Y_{i-1})}{h}$$

$$(iii)\ \frac{\mu h}{2\tau}M_{i-1}=-(\tilde{P}_4m_{i-1}+\tilde{P}_2m_i)+\frac{\tilde{B}_1(Y_i-Y_{i-1})}{h}$$

$$(iv)\frac{\mu h}{2\tau}M_i=(\tilde{P}_3m_{i-1}+\tilde{P}_1m_i)+\frac{\tilde{B}_2(Y_i-Y_{i-1})}{h}$$

where

$$\tilde{P}_1=\frac{1}{4}(1+\mu)+\frac{\mu}{2\tau},\quad \tilde{P}_2=\frac{1}{4}(1-\mu)-\frac{\mu}{2\tau},\quad \tilde{P}_3=\frac{1}{4}(1+\mu)-\frac{\mu}{2\tau},$$
$$\tilde{P}_4=\frac{1}{4}(1-\mu)+\frac{\mu}{2\tau},\quad \tilde{B}_1=\frac{1}{4}(1-\mu),\quad \tilde{B}_2=-\frac{1}{2}(1+\mu),$$

and $\mu=\coth\left(\frac{\tau}{2}\right)-\frac{2}{\tau}$ Also, we get

$$\tilde{P}_2M_{i+1}+\left(\tilde{P}_1+\tilde{P}_4\right)M_i+\tilde{P}_3M_{i-1}=\frac{1}{h^2}\left[Y_{i-1}-2Y_i+Y_{i+1}\right]\tag{5.6}$$

Remark: In the limiting case when $\tau\to 0$, we have
$\widetilde{P}_1=\widetilde{P}_4=\frac{1}{3},\tilde{P}_2=\tilde{P}_3=\frac{1}{6},B_1=\frac{1}{2},B_2=\frac{1}{2},\mu=0,\frac{\mu}{\tau}=\frac{1}{6}$ and the spline function (5.4) reduces to ordinary cubic spline. At the point of grid r_i, Eq.(5.1) may be discretized as

$$-C_\varepsilon\ M_i+\alpha\left(r_i\right)\ Y_i'(v_i)+\widetilde{\beta}\left(r_i\right)\ Y_i(r_i)=G(r_i),\quad 0\le\ i\le\ L\tag{5.7}$$

Introducing a fitting parameter σ in Eq.(5.7) to control the oscillations in solution layer behavior, we get

$$-\sigma\ C_\varepsilon\ M_i+\alpha\left(r_i\right)\ Y_i^{'}(r_i)+\widetilde{\beta}\left(r_i\right)\ Y_i(r_i)=G(r_i),\quad 0\le\ i\le\ L\tag{5.8}$$

with

$$Y\left(0\right)=\phi(0),\ Y\left(1\right)=\psi(1)\tag{5.9}$$

where σ must be chosen so that the solutions of Eqs. (5.8) - (5.9) converge to the solutions of Eqs. (5.1) - (5.2).
Substituting

$$M_i=\frac{1}{\sigma\ C_\varepsilon}\left[\alpha\left(r_i\right)Y_i'\left(r_i\right)+\widetilde{\beta}\left(r_i\right)Y_i\left(r_i\right)-G(r_i)\right]$$

in Eq.(5.6) and using the following first order derivative approximations of Y at the grid points $r_1,r_2,...,r_{L-1}$.

$$Y_{i-1}'\approx\frac{Y_i-Y_{i-1}}{h},\quad Y_i'\approx\frac{Y_{i+1}-Y_{i-1}}{2h}\ ,\quad Y_{i+1}'\approx\frac{Y_{i+1}-Y_i}{h}$$

We get the following tridiagonal system

$$\begin{aligned}&\left(-\sigma C_\varepsilon+\left(\widetilde{\mathrm{P}}_1+\widetilde{\mathrm{P}}_4\right)\frac{h\alpha_i}{2}+\widetilde{\mathrm{P}}_2\left(\alpha_{i+1}h+\widetilde{\beta}_{i+1}h^2\right)\right)\mathrm{Y}_{\mathrm{i+1}}\\&+\left(2\sigma C_\varepsilon+\left(\widetilde{P}_1+\widetilde{P}_4\right)\widetilde{\beta}_i\;h^2\;-\;\widetilde{\mathrm{P}}_2\left(\alpha_{i+1}h\right)+\;\widetilde{\mathrm{P}}_3\left(\alpha_{i-1}h\right)\right)\;\mathrm{Y}_i\\&+\left(-\sigma C_\varepsilon-\frac{h\alpha_i}{2}\left(\widetilde{P}_1+\widetilde{P}_4\right)+\widetilde{P}_3\left(\widetilde{\beta}_{\mathrm{i-1}}h^2-\alpha_{i-1}h\right)\right)\mathrm{Y}_{\mathrm{i-1}}\\&=h^2\left[\widetilde{P}_2G_{i+1}+\left(\widetilde{P}_1+\widetilde{P}_4\right)\;G_i+\widetilde{P}_3G_{i-1}\right],\quad\text{for}\;\;1\le\;i\le\;L-1.\end{aligned}\tag{5.10}$$

$\alpha(r_i)=\alpha_i$, $\widetilde{\beta}(r_i)=\widetilde{\beta}_i$, $\mathrm{G}(r_i)=G_i$ for $i=0,1,\ldots,L$. Using the singular perturbation theory for the layer at the right end of the domain, the solution to the problem represented as of the form [21]

$$Y(r)\approx\;Y_0(r)+\frac{\alpha\;(1)}{\alpha(r)}\left(\psi(1)-Y_0(1)\right)exp\left\{-\frac{\alpha(r)(1-r)}{C_\varepsilon}\right\}+O(C_\varepsilon).\tag{5.11}$$

where $Y_0(r)$ is the solution to the reduced problem. The expansion of Taylor's series is being used for $\alpha(r)$ about a point $v=1$ in Eq.(5.10) and restricting to the first term, we get

$$Y(r)\approx\;Y_0(r)+\left(\psi(1)-Y_0(1)\right)exp\left\{-\frac{\alpha(1)(1-r)}{C_\varepsilon}\right\}+O(C_\varepsilon)\tag{5.12}$$

At $r_i=ih$ and allowing $\mathrm{h}\to0$, we get

$$\lim_{h\to0}Y(ih)\approx\;Y_0(ih)+\left(\psi(1)-Y_0(1)\right)exp\left\{-\alpha(1)\left(\frac{1}{C_\varepsilon}-i\rho\right)\right\}+O(C_\varepsilon),\tag{5.13}$$

where $\rho=\frac{h}{\mathrm{C}_\varepsilon}$ Using Taylor's series and taking the limit as $h\to0$ in Eq.(5.10), we obtain [17]

$$\begin{aligned}&\lim_{h\to0}\left(\frac{\sigma}{\rho}\right)\left[Y(i-1)h-2Y(i)h+Y(i+1)h\right]\\&=\left(\frac{\alpha(0)}{2}\right)\lim_{h\to0}\left[Y(i-1)h-Y(i+1)h\right]\end{aligned}\tag{5.14}$$

Now, substituting Eq.(5.12) into Eq.(5.13) and exercising, we get

$$\sigma=\rho\left(\frac{\alpha(0)}{2}\right)coth\left(\frac{\alpha(1)\rho}{2}\right),\text{where }\rho=\frac{h}{\mathrm{C}_\varepsilon}\tag{5.15}$$

which is a fixed fitting factor in general, we consider the variable fitting parameter to be

$$\sigma=\rho\left(\frac{\alpha(r_i)}{2}\right)coth\left(\frac{\alpha(r_i)\rho}{2}\right)\tag{5.16}$$

Eq.(5.10) can be written as

$$\mathbb{P}_iY_{i-1}+\mathbb{C}_iY_i+\mathbb{Q}_iY_{i+1}=\mathbb{W}_i\;\;\text{for}\;1\le\;i\le\;L-1.\tag{5.17}$$

where

$$\begin{aligned}\mathbb{P}_i&=\left(-\sigma C_\varepsilon-\frac{h\alpha_i}{2}\left(\widetilde{P}_1+\widetilde{P}_4\right)+\widetilde{P}_3\left(\widetilde{\beta}_{\mathrm{i-1}}h^2-\alpha_{i-1}h\right)\right)\\\mathbb{C}_i&=\left(2\sigma C_\varepsilon+\left(\widetilde{P}_1+\widetilde{P}_4\right)\widetilde{\beta}_i\;h^2\;-\;\widetilde{\mathrm{P}}_2\left(\alpha_{i+1}h\right)+\;\widetilde{\mathrm{P}}_3\left(\alpha_{i-1}h\right)\right)\\\mathbb{Q}_i&=\left(-\sigma C_\varepsilon+\left(\widetilde{\mathrm{P}}_1+\widetilde{\mathrm{P}}_4\right)\frac{h\alpha_i}{2}+\widetilde{\mathrm{P}}_2\left(\alpha_{i+1}h+\widetilde{\beta}_{i+1}h^2\right)\right)\\\mathbb{W}_i&=\;h^2\left[\widetilde{P}_2G_{i+1}+\left(\widetilde{P}_1+\widetilde{P}_4\right)\;G_i+\widetilde{P}_3G_{i-1}\right],\end{aligned}$$

The system of equations in Eq (5.16). with boundary conditions Eq (5.8) can be solved by using matrix inverse method and σ is given by Eq (5.15).

6. Error Estimation for the Adaptive Spline Method

This section deals with the ε -uniform convergence of the scheme in(5.17). The local truncation error for $i = 1, 2..., L-1.$ is given by

$$T_i = \mathbb{P}_i Y_{i-1} + \mathbb{C}_i Y_i + \mathbb{Q}_i Y_{i+1} - \mathbb{W}_i$$

Using Taylor series expansion, we get

$$T_i\left(h\right) = \left(\widetilde{P}_3 - \widetilde{P}_2\right)\ h^3\left(\frac{Y_i''}{2}\alpha_i + Y_i'\ \left(\alpha_i' + \widetilde{\beta}\right) + Y_i\widetilde{\beta}_i' - G_i{}'\right)$$

$$+h^4\left[\frac{\widetilde{P}_2 + \widetilde{P}_3}{2} - \frac{1}{6}\right]\left(\sigma C_\varepsilon Y^{iv} - \alpha_i Y_i^{'''}\right) + O\left(h^6\right) \tag{6.1}$$

It is clear that for

$\widetilde{P}_2 = \widetilde{P}_3$, $\frac{\widetilde{P}_2 + \widetilde{P}_3}{2} = \frac{1}{6}$ and $\widetilde{P}_1 + \widetilde{P}_2 + \widetilde{P}_3 + \widetilde{P}_4 = 1$, Eq.(6.1) becomes

$$|T_i(h)| \leq\ K\ O(h^6) \tag{6.2}$$

Incorporating boundary conditions in Eq (5.16), we get a set of equations in the form of a matrix as

$$\left(A + Q\right) W + R + \mathcal{T}_i(h) = 0 \tag{6.3}$$

where $$A = \begin{bmatrix} 2\sigma\varepsilon & -\sigma\varepsilon & 0 & 0 & ... & 0 \\ -\sigma\varepsilon & 2\sigma\varepsilon & -\sigma\varepsilon & .. & ... & 0 \\ 0 & .. & .. & .. & ... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & -\sigma\varepsilon & 2\sigma\varepsilon \end{bmatrix}$$

and

$$Q = [\tilde{z}_i, \tilde{r}_i, \tilde{w}_i] = \begin{bmatrix} \tilde{r}_1 & \tilde{w}_1 & 0 & 0 & .... & 0 \\ \tilde{z}_2 & \tilde{r}_2 & \tilde{w}_2 & 0 & .... & 0 \\ 0 & \tilde{z}_3 & \tilde{r}_3 & \tilde{w}_3 & .... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & \tilde{z}_{L-1} & \tilde{r}_{L-1} \end{bmatrix}$$

where

$$\tilde{z}_i = (-\sigma C_\epsilon + (\tilde{P}_1 + \tilde{P}_4)\frac{ha_i}{2} + \tilde{P}_2(\alpha_{i+1}h + \tilde{\beta}_{i+1}h^2)),$$

$$\tilde{r}_i = (2\sigma C_\epsilon + (\tilde{P}_1 + \tilde{P}_4)\tilde{\beta}_i h^2 - \tilde{P}_2(a_{i+1}h) + \tilde{P}_3(\alpha_{i-1}h)),$$

$$\tilde{w}_i = (-\sigma C_\epsilon - (\tilde{P}_1 + \tilde{P}_4)\frac{ha_i}{2} + \tilde{P}_3(\tilde{\beta}_{i-1}h^2 - \alpha_{i-1}h)) \text{ for } 2 \leq\ i \leq\ L-1.$$

and $R = \left[\tilde{r}_1 - \tilde{z}_1\left(\phi(0)\right), \tilde{r}_2, \tilde{r}_3, ..., \tilde{r}_{L-1}, \tilde{w}_{L-1}\left(\psi(1)\right)\right]^T$,
where

$$\tilde{r}_i = h^2[\tilde{P}_2 G_{i+1} + (\tilde{P}_1 + \tilde{P}_4)G_i + \tilde{P}_3 G_{i-1}], 2 \leqslant i \leqslant L-1$$

$\mathcal{T}_i\left(h\right) = O\left(h^4\right)$ for , $(\tilde{P}_1 + \tilde{P}_4) = \frac{2}{3}$ and $\tilde{P}_2 = \tilde{P}_3 = \frac{1}{6}$, $W = [W_1, W_2, \ldots, W_{L-1}]^{'}$, $\mathcal{T}_i\left(h\right) = [\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{L-1}]^{'}$, $O = [0, 0, \ldots, 0]^T$ are associated vectors of Eq. (6.3).

Let $\omega = [\omega_1, \omega_2, ..., \omega_{L-1}]^{'} \cong W$ satisfies the equation

$$\left(A + Q\right)\omega + R = 0 \tag{6.4}$$

Let $e_i = \omega_i - W_i$, $1 \leq \ i \leq \ L-1$, be the discretized error $E = [e_1, e_2, ..., e_{L-1}]^{'} = \omega - W$. Using Eq.(6.3) and Eq. (6.4), we get the error equation as

$$(A+Q)\,E = \mathcal{T}_i(h) \tag{6.5}$$

Let $|\alpha(r)| \leq \eta_1$ and $|\beta\,(r)| \leq \eta_2$, where η_1, η_2 are positive constants. Let $(i,j)^{th}$ element of the matrix $(A+Q)$ be $\zeta_{i,j}$ then

$$|\zeta_{i,i+1}| \leqslant [-\sigma C_\epsilon - (\tilde{P}_1 + \tilde{P}_4)\frac{ha_i}{2} + \tilde{P}_3(\beta_{i-1}h^2 - \alpha_{i-1}h)] \text{ for } 2 \leqslant i \leqslant L-2$$

$$|\zeta_{i,i-1}| \leqslant [-\sigma C_\epsilon + (\tilde{P}_1 + \tilde{P}_4)\frac{ha_i}{2} + \tilde{P}_2(\alpha_{i+1}h + \beta_{i+1}h^2)] \text{ for } 2 \leqslant i \leqslant L-1.$$

Hence, for small h, we have

$$|\zeta_{i,i+1}| < \sigma C_\varepsilon, \quad 1 \leqslant i \leqslant L-2.$$

and

$$|\zeta_{i,i-1}| < \sigma C_\varepsilon, \quad 2 \leqslant i \leqslant L-1. \tag{6.6}$$

Let $\mathrm{S_i}$ be the i^{th} row elements sum, of the matrix $(A+Q)$ be, then we have

$$S_i = [\sigma C_\epsilon + h((\tilde{P}_1 + \tilde{P}_4)\frac{a_i}{2} + \tilde{P}_3\alpha_{i-1}) + h^2((\tilde{P}_1 + \tilde{P}_4)\beta_i + \tilde{P}_2\beta_{i+1})], \text{for} \ \ i=1.$$

$$S_i = h^2[\tilde{P}_3\beta_{i-1} + (\tilde{P}_1 + \tilde{P}_4)\beta_i + \tilde{P}_3\beta_{i+1}], \text{for} \ \ 2 \leq \ i \leq \ L-2.$$

$$S_i = [\sigma C_\epsilon + h(-(\tilde{P}_1 + \tilde{P}_4)\frac{a_i}{2} - \tilde{P}_2\alpha_{i+1}) + h^2((\tilde{P}_1 + \tilde{P}_4)\beta_i + \tilde{P}_3\beta_{i-1})] \ \text{ for } \ i = L-1.$$

Let $\xi_{1^*} = \min\limits_{1 \leq i \leq L-1} |\alpha(r)|$ and $\xi_1^* = \min\limits_{1 \leq i \leq L-1} |\alpha(r)|$, $\xi_{2^*} = \min\limits_{1 \leq i \leq L-1} |\beta(r)|$ and $\xi_2^* = \min\limits_{1 \leq i \leq L-1} |\beta(r)|$. Since $0 < C_\varepsilon \ll 1$, and $C_\varepsilon \propto O\,(h)$ it is verified that for sufficiently small h. $(A+Q)$ is monotone. Hence$(A+Q)^{-1}$ exists and $(A+Q)^{-1} \geqslant 0$. Thus using Eq. (6.5), we have

$$\|E\| \leqslant \left\|(A+Q)^{-1}\right\| \|\mathcal{T}_i\| \tag{6.7}$$

Let $(A+Q)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(A+Q)^{-1}$ and define $\|\,(A+Q)^{-1}\,\| = \max\limits_{1 \leq i \leq L-1} \sum_{k=1}^{L-1} (A+Q)^{-1}_{i,k}$ and $\|\mathcal{T}_i(h)\| = \max\limits_{1 \leq i \leq L-1} |T_i(h)|$

$$\text{Since } \ (A+Q)^{-1}_{i,k} \geqslant 0 \ \text{ and } \sum_{k=1}^{L-1} (A+Q)^{-1}_{i,k}.S_k = 1, \text{ for } 1 \leqslant i \leqslant L-1, \tag{6.8}$$

we have,

$$(A+Q)^{-1}_{i,k} \leqslant \frac{1}{S_1} < \frac{1}{h^2[\tilde{P}_2\beta_{i+1} + (\tilde{P}_1 + \tilde{P}_4)\beta_i]} \ , \ i=1, \tag{6.9}$$

$$(A+Q)^{-1}_{i,k} \leqslant \frac{1}{S_{L-1}} < \frac{1}{h^2[\tilde{P}_3\beta_{i-1} + (\tilde{P}_1 + \tilde{P}_4)\beta_i]} \ , \ i=L-1, \tag{6.10}$$

Further

$$\sum_{k=1}^{L-1} (A+Q)^{-1}_{i,k} \leqslant \frac{1}{\min\limits_{2 \leq i \leq L-2} S_k} \tag{6.11}$$

$$< \frac{1}{h^2[\tilde{P}_3\beta_{i-1} + (\tilde{P}_1 + \tilde{P}_4)\beta_i + \tilde{P}_2\beta_{i+1}]} \ , \text{ for } 2 \leqslant i \leqslant L-2.$$

From error Eq. (6.7)using of Eqs. (6.8) - (6.11) we get

$$\|E\| \leqslant O\left(h^2\right) \tag{6.12}$$

method given by Eq. (5.15) is fourth order convergent in space and first order in time for $(\tilde{P}_1 + \tilde{P}_4) = \frac{2}{3}$ and $\tilde{P}_2 = \tilde{P}_3 = \frac{1}{6}$

Lemma 6.1 *Let $W(r,t_j)$ be the solution of the problem (6.1) and w_i be an computational solution to $W(r,t_j)$ of Eq. (6.11) after the time discretization. Then the error estimate in the totally discrete scheme is $max_{0<C_\varepsilon\ll 1}\|W(r,t_n)-w(r,t_n)\|_\infty \leq C(\Delta t+L^{-4})$*

Proof. From triangular inequality we have

$$max_{0<C_\varepsilon\ll 1\|W(r,t_n)-w(r,t_n)}\|_\infty \leq maxW(r,t_n)-w(r,t_n)+w(r,t_n)-W(r,t_n)$$

$$\leq max\,|W(r,t_n)-\ w(r,t_n)+w(r,t_n)-\ W(r,t_n)|$$

Using Lemma 4.1 and 4.3, we get the required estimate. Therefore,

$$max_{0<C_\varepsilon\ll 1\|W(r,t_n)-w(r,t_n)\|_\infty} \leq C(\Delta t+L^{-4}).$$

7. Numerical Experiments

In order to illustrate the applicability and competence of the numerical approach presented for the problem (2.1), we considered two numerical experiments that are widely used in the literature. Because the exact solutions to these problems are unknown, the maximum point-wise errors are presented in tabular form using the double mesh principle defined by $ER^{L,M}_{\varepsilon,\delta,\eta}=\max_{1\leq\ i\leq\ L}\left|Y^{L,M}_i-Y^{2L,2M}_{2i}\right|$,

Example 1.

$$\frac{\partial v}{\partial t}-\varepsilon^2\frac{\partial^2 v}{\partial z^2}+\frac{\partial v}{\partial z}+\left(2\ -\ z^2\right)v\left(z-\delta,t\right)+\left(1+z^2\right)v\left(z,t\right)+e^z v\left(z+\eta,t\right)$$

$$=50(z(1-z))^3 where$$

$$v_0=0,\phi_1\left(z,t\right)=0,\phi_2\left(z,t\right)=0,\text{with } T=2.$$

Example 2.

$$\frac{\partial v}{\partial t}-\varepsilon^2\frac{\partial^2 v}{\partial z^2}+\left(2-z^2\right)\frac{\partial s}{\partial z}-\left(1+z\right)v\left(z-\delta,t\right)+\ \left(1+z^2+\cos\left(\pi z\right)\right)v\left(z,t\right)$$

$$+3y\left(z+\eta,t\right)=\sin(\pi z)$$

with

$$\left(z,t\right)\in\left(0,1\right)\times\left(0,1\right),v(z,t)=0 \text{ for } \left(z,t\right)\in\left(-\delta,0\right)\times\left(0,1\right]$$

$$\text{and }\ v(z,t)=0 \text{ for } \left(z,t\right)\in\left(1,1+\eta\right)\times\left(0,1\right]$$

8. Conclusion

A fitted adaptive spline computational scheme for time dependent SPPDDE with small retarded terms in the reaction terms is presented. We approximated the delay and advance terms using Taylor's series expansion when δ and η are of the order of $o\left(\varepsilon\right)$. The resulting equation is solved on a uniform mesh using the backward Euler method for discretizing temporal variables and adaptive cubic splines with fitting parameter for spatial discretization. The error estimation for the proposed scheme is presented. To validate the superiority of the scheme, we calculated the maximum point-wise errors for the numerical experiments under consideration. In Tables 1 and 2, it is observed that the point-wise errors decrease as $\varepsilon\to 0$ and the grid size decreases, which shows that the proposed method is ε-uniformly convergent and stable. Also, we presented the numerical solution profile of the examples for various values of ε, δ, η and L.

9. Tables and Figures

Table 1. Maximum AEs in Example 1 with $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$.

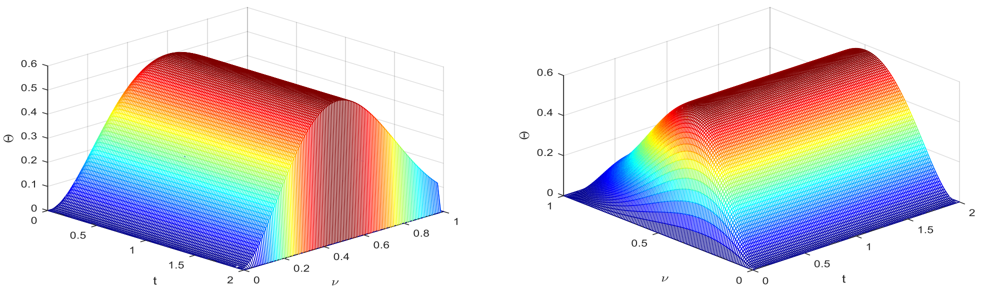
$\varepsilon \downarrow$	$L = 2^6$ $\Delta r = 0.125$	$L = 2^7$ $\Delta r = \frac{0.125}{2}$	$L = 2^8$ $\Delta r = \frac{0.125}{2^2}$	$L = 2^9$ $\Delta r = \frac{0.125}{2^3}$	$L = 2^{10}$ $\Delta r = \frac{0.125}{2^4}$
suggested method					
2^{-6}	2.9371(-4)	2.3241(-4)	1.5968(-4)	8.7303(-5)	2.4562(-5)
2^{-8}	7.9429(-4)	2.9084(-4)	2.2467(-4)	2.0821(-4)	1.8420(-4)
2^{-10}	7.2539(-4)	1.8434(-4)	4.9940(-4)	1.8527(-4)	1.4816(-4)
2^{-12}	7.2098(-4)	1.7996(-4)	4.5176(-4)	1.1501(-4)	3.1221(-5)
2^{-14}	7.2071(-4)	1.7970(-4)	4.4916(-4)	1.1337(-4)	2.8214(-5)
2^{-16}	7.2069(-4)	1.7969(-4)	4.4900(-4)	1.1321(-4)	2.8046(-5)
2^{-18}	7.2069(-4)	1.7969(-4)	4.4899(-4)	1.1320(-4)	2.8046(-5)
Results in [16]					
2^{-6}	7.0951(-3)	4.3731(-3)	2.4760(-3)	1.3268(-3)	6.8848(-4)
2^{-8}	7.3639(-3)	4.5420(-3)	2.5915(-3)	1.3942(-3)	7.2395(-4)
2^{-10}	7.4542(-3)	4.5941(-3)	2.6318(-3)	1.4161(-3)	7.3588(-4)
2^{-12}	7.4772(-3)	4.6091(-3)	2.6447(-3)	1.4234(-3)	7.3998(-4)
2^{-14}	7.4834(-3)	4.6157(-3)	2.6491(-3)	1.4262(-3)	7.4151(-4)
2^{-16}	7.4853(-3)	4.6182(-3)	2.6508(-3)	1.4273(-3)	7.4214(-4)
2^{-18}	7.4860(-3)	4.6192(-3)	2.6516(-3)	1.4278(-3)	7.4242(-4)

Table 2. Maximum AEs in Example 2 with $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$.

$\varepsilon \downarrow$	$L = 2^4$ $K = 30$	$L = 2^5$ $K = 60$	$L = 2^6$ $K = 120$	$L = 2^7$ $K = 240$	$L = 2^8$ $K = 480$	$L = 2^9$ $K = 960$
suggested method						
2^{-8}	3.4763(-3)	1.3455(-3)	3.6782(-4)	8.1582(-5)	3.1274(-5)	2.1482(-6)
2^{-10}	3.4863(-3)	1.3648(-3)	3.8068(-4)	8.4513(-5)	2.3613(-5)	5.8244(-5)
2^{-12}	3.4876(-3)	1.3726(-3)	3.8868(-4)	8.9274(-5)	2.4529(-5)	5.9143(-5)
2^{-14}	4.4877(-3)	1.3751(-3)	3.8918(-4)	8.9689(-4)	2.5179(-5)	6.2865(-5)
2^{-16}	4.4877(-3)	1.3730(-3)	3.8921(-4)	8.9722(-4)	2.5102(-5)	6.3206(-5)
2^{-20}	4.4877(-3)	1.3730(-3)	3.8921(-4)	8.9744(-4)	2.5105(-5)	6.3229(-5)
Results in [10]						
2^{-8}	1.5212(-2)	7.6301(-3)	3.8340(-3)	1.9258(-3)	9.6540(-4)	4.8334(-4)
2^{-10}	1.5227(-2)	7.6372(-3)	3.8376(-3)	1.9273(-3)	9.6612(-4)	4.8371(-4)
2^{-12}	1.5230(-2)	7.6384(-3)	3.8381(-3)	1.9276(-3)	9.6623(-4)	4.8377(-4)
2^{-14}	1.5230(-2)	7.6386(-3)	3.8382(-3)	1.9276(-3)	9.6626(-4)	4.8377(-4)
2^{-16}	1.5240(-2)	7.6387(-3)	3.8383(-3)	1.9276(-3)	9.6626(-4)	4.8378(-4)
2^{-20}	1.5240(-2)	7.6387(-3)	3.8383(-3)	1.9276(-3)	9.6626(-4)	4.8378(-4)

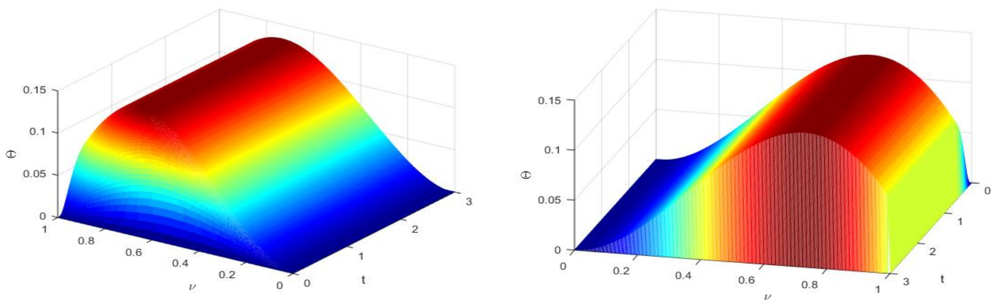
Table 3. Maximum AEs in Example 3 with $\delta = \eta = 0.5\varepsilon$.

$\varepsilon \downarrow$	$L = M$ $L = 2^5$	$L = 2^6$	$L = 2^7$	$L = 2^8$	$L = 2^9$	$L = 2^{10}$
suggested method						
10^{-1}	3.3641(-3)	2.3404(-3)	9.9611(-4)	2.6702(-4)	6.9937(-6)	4.7637(-6)
10^{-2}	1.4269(-3)	3.4805(-4)	9.9061(-4)	1.8110(-5)	8.8171(-6)	2.1134(-6)
10^{-3}	1.4872(-3)	3.7670(-4)	9.5233(-4)	2.3810(-5)	5.9453(-6)	1.4777(-6)
10^{-4}	1.4778(-3)	3.7900(-4)	9.5179(-4)	2.3881(-5)	5.9811(-5)	1.4955(-5)
10^{-5}	1.4878(-3)	3.7900(-4)	9.5180(-4)	2.3882(-5)	5.9812(-5)	1.4957(-5)
10^{-6}	1.4878(-3)	3.7900(-4)	9.5180(-4)	2.3882(-5)	5.9804(-5)	1.4957(-5)
10^{-8}	1.4878(-3)	3.7900(-4)	9.5180(-4)	2.3882(-5)	5.9804(-5)	1.4957(-5)
Results in [8]						
10^{-1}	7.9508(-3)	4.8116(-3)	2.7152(-3)	1.4522(-3)	7.5168(-4)	3.8219(-4)
10^{-2}	8.0520(-3)	5.0091(-3)	2.8755(-3)	1.5525(-3)	8.0567(-4)	4.1008(-4)
10^{-3}	8.0491(-3)	5.0051(-3)	2.8716(-3)	1.5497(-3)	8.0408(-4)	4.0925(-4)
10^{-4}	8.0488(-3)	5.0047(-3)	2.8712(-3)	1.5494(-3)	8.0392(-4)	4.0917(-4)
10^{-5}	8.0488(-3)	5.0046(-3)	2.8712(-3)	1.5494(-3)	8.0390(-4)	4.0916(-4)
10^{-6}	8.0488(-3)	5.0046(-3)	2.8712(-3)	1.5494(-3)	8.0390(-4)	4.0916(-4)
10^{-8}	8.0488(-3)	5.0046(-3)	2.8712(-3)	1.5494(-3)	8.0390(-4)	4.0916(-4)



(i) (ii)

Fig. 1. Layer Profile in Ex 1 for $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$, $N = K = 128$, (i) $\varepsilon = 2^{-6}$, (ii) $\varepsilon = 2^{-12}$



(i) (ii)

Fig. 2. Layer Profile in Ex 2 with $\delta = \eta = 0.5\varepsilon$, $K = N = 264$, (i) $\varepsilon = 2^{-6}$, (ii) $\varepsilon = 2^{-8}$.

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