Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 17** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.82419

Robust Fitted Parameter Finite Difference Method for Delay-Influenced Singularly Perturbed Differential-Difference Equations

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ABSTRACT: In this study, we present a novel computational method based on the finite difference approach to solve a class of differential–difference equations characterized by a negative shift in the differentiated term. When the shift parameter is of order $O(\varepsilon)$, the proposed scheme demonstrates strong performance and effectively suppresses oscillations in the solution's boundary-layer region. To achieve this, we introduce a parameter into the numerical scheme, constructed on a specially designed mesh, whose evaluation is guided by the theory of singular perturbations. Numerical experiments are conducted to validate the method, with maximum absolute errors and convergence orders tabulated to illustrate its efficiency and to substantiate the theoretical convergence analysis.

Keywords: Singularly perturbed problems, delay differential equation, boundary layer, exponential spline.

Contents

1. Introduction

Differential- difference equations are problems in which the time evolution of the state variable can depend on specific past in some arbitrary way, i.e., the rate of physical system change depends not only on the state of the physical system but also on its history [2]. Such problems also occur when many practical phenomena are modelled such as thermo-elasticity [3], hybrid optical system [4], in population dynamics [15], red blood cell system [19], in models for physiological processes [21], predator-prey models [22], optimal control theory [8] and in the potential in nerve cells by synaptic inputs in dendrites [26]. Further analysis on mathematical aspects of the above group of modelling problems and of the singular perturbations is available from the collection of books to name but a few, Bellman and Cooke [2], Doolan et al. [5], Driver [6], El'sgol'ts and Norkín [7], Kokotovic et al. [14], Miller et al. [26] and Smith [32], neuronal variability problems connected to patterns of nerve action potentials formed by unit quantal inputs occurring at random are studied [33]. We investigated the influence of large delays on the layer structure

2020 *Mathematics Subject Classification:* 65L11, 65L12.
Submitted March 04, 2026. Published June 19, 2026.

and oscillatory behaviour of solutions by employing a specially designed mesh, both with and without fitting parameter[1,20,31]. The authors in[16,23,24,25,35] investigated singularly perturbed differential equations incorporating delays in both convection and reaction terms, and developed numerical schemes tailored to address the associated stiffness and boundary layer behavior. The authors in [17, 18] provided asymptotic access for a class of layer behaviour differential equations. In [26], researchers provided a mathematical model of evaluating the expected time to produce potential action in nerve cells with random synaptic inputs in dendrites. The researchers in [27,30] illustrated the examples with quick oscillations. Kadalbajoo and his team initiated an extensive numerical work using finite differentiation techniques [10, 11, 12, 13]. Ravikanth and Murali [29] have proposed a fitted method to solve the problems through tension splines, which only contains a delay in the differentiated term.

2. Statement of the Problem

Here we have considered a singularly perturbed differential-difference equation (SPDDEs) with a delay term, that is, a negative shift is of the form

$$L_{\varepsilon,\delta}\big(\nu(s)\big) \equiv \varepsilon Z''(s) + P(s)\,Z'(s-\delta) + R(s)\,Z(s-\delta) + B(s)\,Z(s) = G, \quad 0 \le s \le 1. \tag{2.1}$$

in accordance with the conditions

$$Z(s) = \mu(s), \quad -\delta \le s \le 0, \qquad Z(1) = \gamma. \tag{2}$$

Here, $0 < \varepsilon \ll 1$, and the functions $p(s), q(s), r(s)$, and $\phi(s)$ are assumed to be smooth and bounded in the interval $[0,1]$. The parameter δ denotes the delay, while ϕ is a finite constant. When $\delta = 0$, Eq. (1) reduces to a singularly perturbed equation exhibiting boundary layer behaviour and turning points that depend on the sign of $p(s)$. Throughout the interval $[0,1]$, the solution $\nu(s)$ develops a boundary layer on the left-hand side if $p(s) > 0$, and on the right-hand side if $p(s) < 0$. When the delay parameter $\delta(\varepsilon)$ is of order $\mathcal{O}(\varepsilon)$, the behaviour of the boundary layer can change significantly, and may even be destroyed, or the solution may exhibit oscillatory behaviour. To monitor these oscillations in the solution profile, we devised a new method that combines a special mesh introduced in [21] with the fitting parameter of the exponential spline method.

3. Numerical Method Using an Exponential Spline

The region $[p,q]$ is subdivided into N equal subregions of mesh size $h = \frac{q-p}{N}$, so that$\theta_i = p+ih$, $\quad i = 0,1,\ldots,N$ are the mesh points. To manage the delay term, choose the mesh size as $h = \frac{\delta}{m}$, $\quad m = \omega l$, where Q is a positive integer and R is the mantissa of δ. Let $z(u)$ be the exact solution to the problem in Eq. (1), and let u_i be an approximate solution to $z(u_i)$ obtained by the segment $Q_i(u)$ passing through the points (u_i, Q_i) and (u_{i+1}, Q_{i+1}). Each exponential spline segment $Q_i(u)$ has the form [9,28]: $Q_i(u) = \cdots$

$$Q_i(u) = a_i e^{k(u-u_i)} + b_i e^{-k(u-u_i)} + c_i(u-u_i) + d_i, \qquad i = 0,1,2,\ldots,n. \tag{3.1}$$

Here a_i, b_i, c_i, and d_i are constants, and k is a free parameter. To find the values of the coefficients in Eq. (3), the interpolatory conditions

$$Q_i(u_i) = S_i, \qquad Q_i(u_{i+1}) = S_{i+1}, \qquad Q_i''(u_i) = M_i, \qquad Q_i''(u_{i+1}) = M_{i+1}$$

are used.Using these conditions, we obtain the following expressions:

$$a_i = \frac{h^2\left(M_{i+1} - e^{-\theta} M_i\right)}{2\theta^2 \sinh(\theta)}, \qquad b_i = \frac{h^2\left(M_i - e^{-\theta} M_{i+1}\right)}{2\theta^2 \sinh(\theta)},$$

$$c_i = \frac{S_{i+1} - S_i}{h} - \frac{h\left(M_{i+1} - M_i\right)}{\theta^2}, \qquad d_i = S_i - \frac{h^2 M_i}{\theta^2},$$

where $\theta = kh$ and $i = 0,1,2,\ldots,n$. By enforcing the first derivative continuity condition

$$Q_{i-1}'(u) = Q_i'(u) \quad \text{at the point } (u_i, Q_i),$$

we obtain the following relation:

$$(z_{i+1} - 2z_i + z_{i-1}) = h^2 \left(\alpha M_{i+1} + \beta M_i + \alpha M_{i-1}\right), \qquad i = 0, 1, \ldots, n-1. \tag{3.2}$$

where

$$\alpha = \frac{\sinh(\theta) - \theta}{\theta^2 \sinh(\theta)}, \qquad \beta = \frac{2\theta\cosh(\theta) - 2\sinh(\theta)}{\theta^2 \sinh(\theta)}.M_j = Z''(s_j), \quad j = i, i \pm 1.$$

4. Numerical Approach with Fitting Parameter for Large Delay

The solution's layer behaviour is preserved and precise results are obtained when, compared to the perturbation parameter, the shift parameter is smaller [30]. However, if $\delta(\varepsilon)$ is of order $\mathcal{O}(\varepsilon)$, the solution's layer behaviour is no longer well-preserved, and oscillations become visible. As a result, we are developing a numerical scheme with fitting parameters that is based on an exponential spline method and a special type of mesh developed in [20]. In addition, we examine the solution layer behaviour graphically for large delays in order to demonstrate the significance of the fitting parameter in our proposed scheme.

4.1. Left-layer numerical scheme

Assume that $P(s) \geq \overline{M}$ and $\varepsilon \geq 0$ in $[0, 1]$, where $\overline{M}$ is a positive constant. As a result of this theory, Eqs. (1)-(2) shows layer behaviour at s=0 for small values of ε. Now, we consider a mesh developed in [21], i.e., chooses the mesh as $h = \delta/m$, where $m = \omega l$, l is the mantissa of δ and ω is positive integer. The discretization of the boundary value problem Eq. (1)-(2) with a special mesh result in

$$\varepsilon M_j = G(s_i) - P(s_i)Z'_{i-m} - R(s_i)Z_{i-m} - B(s_i)Z_i, \forall M_j, \quad j = i, i \pm 1. \tag{4.1}$$

with

$$Z_i = \phi_i \quad \text{for } i = -m, -m+1, \ldots, 0, \qquad Z_N = \phi \tag{4.2}$$

The difference approximations for the first-order derivatives listed below,

$$Z'_{i-m-1} = \frac{-Z_{i-m+1} + 4Z_{i-m} - 3Z_{i-m-1}}{2h} + O(h^2)$$

$$Z'_{i-m} = \frac{Z_{i-m+1} - Z_{i-m-1}}{2h} + O(h^2)$$

$$Z'_{i-m+1} = \frac{3Z_{i-m+1} - 4Z_{i-m} + Z_{i-m-1}}{2h} + O(h^2).$$

By applying the difference scheme with the values of M_{i-1},M_i and M_{i+1} are substituted in Eq. (3.2), Eq. (4.1) and together with the difference approximations for the first-order derivatives then the difference scheme is

$$\begin{gathered}(\varepsilon + \alpha h^2 B_{i-1})Z_{i-1} + \left(-2\varepsilon + \beta h^2 B_i\right)Z_i + \left(\varepsilon + \alpha h^2 B_{i+1}\right)Z_{i+1} + \\ \left(-\tfrac{\alpha h P_{i-1}}{2} + \tfrac{\beta h P_i}{2} + \tfrac{3\alpha h P_{i+1}}{2} + \alpha R_{i+1}\right) Z_{i-m+1} + \left(2\alpha h P_{i-1} - 2\alpha h P_{i+1} + \beta R_i\right) Z_{i-m} + \\ \left(-\tfrac{3\alpha h P_{i-1}}{2} - \tfrac{\beta h P_i}{2} + \tfrac{\alpha h P_{i+1}}{2} + \alpha R_{i+1}\right) Z_{i-m-1} = h^2 \left(\alpha G_{i-1} + \beta G_i + \alpha G_{i+1}\right).\end{gathered} \tag{4.3}$$

Utilizing Eq. (4.3), the difference scheme is transformed

$$S_i Z_{i-1} + G_i Z_i + F_i Z_{i+1} + A_i Z_{i-m+1} + X_i Z_{i-m} + w_i Z_{i-m-1} = G_i, \quad i = 1, 2, \ldots, N-1. \tag{4.4}$$

where

$$S_i = \varepsilon + \alpha h^2 B_{i-1}, G_i = -2\varepsilon + \beta h^2 B_i, F_i = \varepsilon + \alpha h^2 B_{i+1},$$

$$A_i = -\tfrac{\alpha h P_{i-1}}{2} + \tfrac{\beta h P_i}{2} + \tfrac{3\alpha h P_{i+1}}{2} + \alpha R_{i+1}, X_i = 2\alpha h P_{i-1} - 2\alpha h P_{i+1} + \beta R_i,$$

$$w_i = -\tfrac{3\alpha h P_{i-1}}{2} - \tfrac{\beta h P_i}{2} + \tfrac{\alpha h P_{i+1}}{2} + \alpha R_{i+1}, H_i = h^2 \left(\alpha G_{i-1} + \beta G_i + \alpha G_{i+1}\right).$$

By utilizing the boundary condition's Eq. (4.2), the approach Eq. (4.4) can be expressed as

$$E_{i-1}\,\omega_{i-1}+V_i\,\omega_i+F_{i+1}\,\omega_{i+1}=H_i-D_i\,\omega_{i-m+1}-Y_i\,\omega_{i-m}-Q_i\,\omega_{i-m-1},1\le i\le m-1$$
$$E_{i-1}\,\omega_{i-1}+V_i\,\omega_i+F_{i+1}\,\omega_{i+1}+D_i\,\omega_{i-m+1}=H_i-Y_i\,\omega_{i-m}-Q_i\,\omega_{i-m-1},i=m$$
$$E_{i-1}\,\omega_{i-1}+V_i\,\omega_i+F_{i+1}\,\omega_{i+1}+D_i\,\omega_{i-m+1}+Y_i\,\omega_{i-m}=H_i-Q_i\,\omega_{i-m-1},i=m+1$$

$$E_{i-1}\,\omega_{i-1}+V_i\,\omega_i+F_{i+1}\,\omega_{i+1}+D_i\,\omega_{i-m+1}+Y_i\,\omega_{i-m}+Q_i\,\omega_{i-m-1}=H,m+2\le i\le N-1 \tag{4.5}$$

When the delay dominates the perturbation, the boundary layer may undergo a qualitative change in its structure, potentially being demolished or even eliminated. In such cases, the solution may exhibit oscillatory behavior. By reorganizing the preceding equation (4.3), we obtain

$$\frac{\varepsilon}{h^2}\left(Z_{i+1}-2Z_i+Z_{i-1}\right)=\left(\frac{-\alpha P_{i+1}}{2h}+\frac{\beta P_i}{h}+\frac{3\alpha P_{i-1}}{2h}-\alpha R_{i-1}\right)Z_{i-m-1}+$$
$$\left(\frac{2\alpha P_{i+1}}{h}-\frac{2\alpha P_{i-1}}{h}-\beta R_i\right)Z_{i-m}+\left(\frac{-3\alpha P_{i+1}}{2h}-\frac{\beta P_i}{h}+\frac{\alpha P_{i-1}}{2h}-\alpha R_{i+1}\right)Z_{i-m+1}$$
$$-\alpha B_{i-1}Z_{i-1}-\beta B_iZ_i-\alpha B_{i+1}Z_{i+1}+\left(\alpha G_{i+1}+\beta G_i+\alpha G_{i-1}\right).$$

To handle this layer character, we tried by introducing a fitting parameter in this scheme. The fitting factor is determined with the help of theory of singular perturbations. Inserting the fitting factor $\sigma(\rho)$ into the above scheme, we obtain

$$\frac{\sigma(\rho)\,\varepsilon}{h^2}\left(Z_{i+1}-2Z_i+Z_{i-1}\right)=\left(\frac{-\alpha P_{i+1}}{2h}+\frac{\beta P_i}{h}+\frac{3\alpha P_{i-1}}{2h}-\alpha R_{i-1}\right)Z_{i-m-1}$$
$$\left(\frac{2\alpha P_{i+1}}{h}-\frac{2\alpha P_{i-1}}{h}-\beta R_i\right)Z_{i-m}+\left(\frac{-3\alpha P_{i+1}}{2h}-\frac{\beta P_i}{h}+\frac{\alpha P_{i-1}}{2h}-\alpha R_{i+1}\right)Z_{i-m+1}$$
$$-\alpha B_{i-1}Z_{i-1}-\beta B_iZ_i-\alpha B_{i+1}Z_{i+1}+\left(\alpha G_{i+1}+\beta G_i+\alpha G_{i-1}\right),\quad \forall i=1,2,\ldots,N-1 \tag{4.6}$$

By multiplying above Equation by h and assuming the limit as $h\to 0$ [4], we obtain

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(Z_{i+1}h-2Z_ih+Z_{i-1}h\right)=\left(\alpha+\tfrac{\beta}{2}\right)P(0)\lim_{h\to 0}\left(Z_{i-m-1}h-Z_{i-m+1}h\right) \tag{4.7}$$

Based on the principles of singular perturbation theory, particularly in relation to the boundary layer at the left end of the interval [16], we obtain

$$\lim_{h\to 0}Z(ih)\approx Z_0(ih)+\big(\phi(0)-Z_0(0)\big)\exp\{-P(0)\,i\rho\}+O(\varepsilon),\quad \text{where } \rho=\tfrac{h}{\varepsilon}. \tag{4.8}$$

By substituting Eq. (4.8) into Eq. (4.6) and carrying out the necessary simplifications, we arrive at

$$\sigma=\rho\left(\alpha+\tfrac{\beta}{2}\right)p(0)\,e^{P(0)m\rho}\,\coth\!\left(\tfrac{P(0)\rho}{2}\right) \tag{4.9}$$

which is the parameter that has been fitted for the layer on the left.

4.2. Right-layer numerical scheme

Assume that $P(s)\le\overline{M}<0$ and $\varepsilon\ge 0$ in $[0,1]$, where $\overline{M}$ is a positive constant. As a result of this theory, Eqs. (1)–(2) exhibit layer behaviour at $s=1$ for small values of ε. Based on singular perturbation theory for the boundary layer at the right end of the interval [16], we have

$$\lim_{h\to 0}Z(ih)=Z_0(0)+\big(\phi(0)-Z_0(1)\big)e^{-P(1)\left(\frac{1}{\varepsilon}-i\rho\right)}+O(\varepsilon),\quad \text{where } \rho=\tfrac{h}{\varepsilon}. \tag{4.10}$$

In the exponential spline difference scheme given by Eq. (4.3), we introduce a fitting parameter $\sigma(\rho)$ and obtain the following expression for the fitting parameter:

$$\sigma = \rho\left(\alpha + \tfrac{\beta}{2}\right) P(0)\, e^{-P(0)m\rho} \coth\left(\tfrac{P(1)\rho}{2}\right) \tag{4.11}$$

By employing a procedure analogous to that applied in the analysis of the left boundary layer,and upon simplification of the difference scheme in Eq. (4.6), we arrive at

$$S_i Z_{i-1} + G_i Z_i + F_i Z_{i+1} + A_i Z_{i-m+1} + X_i Z_{i-m} + w_i Z_{i-m-1} = G_i, \quad i = 1, 2, \ldots, N-1. \tag{4.12}$$

where

$$S_i = \sigma\varepsilon + \alpha h^2 B_{i-1}, G_i = -2\sigma\varepsilon + \beta h^2 B_i, F_i = \sigma\varepsilon + \alpha h^2 B_{i+1},$$

$$A_i = -\tfrac{\alpha h P_{i-1}}{2} + \tfrac{\beta h P_i}{2} + \tfrac{3\alpha h P_{i+1}}{2} + \alpha Q_{i+1}, X_i = 2\alpha h P_{i-1} - 2\alpha h P_{i+1} + \beta Q_i,$$

$$w_i = -\tfrac{3\alpha h P_{i-1}}{2} - \tfrac{\beta h P_i}{2} + \tfrac{\alpha h P_{i+1}}{2} + \alpha Q_{i+1}, H_i = h^2\left(\alpha G_{i-1} + \beta G_i + \alpha G_{i+1}\right).$$

By utilizing the boundary condition's Eq. (4.2), the approach Eq. (4.12) can be expressed as

$$E_{i-1}\,\omega_{i-1} + V_i\,\omega_i + F_{i+1}\,\omega_{i+1} = H_i - D_i\,\omega_{i-m+1} - Y_i\,\omega_{i-m} - R_i\,\omega_{i-m-1}, 1 \le i \le m-1$$

$$E_{i-1}\,\omega_{i-1} + V_i\,\omega_i + F_{i+1}\,\omega_{i+1} + D_i\,\omega_{i-m+1} = H_i - Y_i\,\omega_{i-m} - R_i\,\omega_{i-m-1}, i = m$$

$$E_{i-1}\,\omega_{i-1} + V_i\,\omega_i + F_{i+1}\,\omega_{i+1} + D_i\,\omega_{i-m+1} + Y_i\,\omega_{i-m} = H_i - R_i\,\omega_{i-m-1}, i = m+1$$

$$E_{i-1}\,\omega_{i-1} + V_i\,\omega_i + F_{i+1}\,\omega_{i+1} + D_i\,\omega_{i-m+1} + Y_i\,\omega_{i-m} + Q_i\,\omega_{i-m-1} = H, m+2 \le i \le N-1 \tag{4.13}$$

5. Truncation Error

The local truncation error for the scheme (4.12) is obtained using Taylor's series expansion about w_{i-m} on Eq. (4.13) and with Eq. (4.12), as shown below:

$$T_i(h) = \sigma\varepsilon\big[1 - 2(\alpha+\beta)\big]h^2\, Z_i^{(2)}(\xi_i) + \left\{\frac{\sigma\varepsilon}{12}\big[1 - 12\alpha\big]\, Z_i^{(4)}(\xi_i) + \frac{1}{3}\big[-2\alpha + \beta\big]P_i(\xi_i)\, Z_i^{(3)}(\xi_i)\right\} h^4 + \mathcal{O}(h^6), \tag{5.1}$$

$$s_i \le \xi_i \le s_{i+1}.$$

Clearly,

$$T_i(h) = \mathcal{O}(h^4)$$

for any random choice of α and β such that $\alpha + \beta = \frac{1}{2}$ and $\alpha = \frac{1}{12}$.

6. Convergence Analysis

The matrix equation for the left-end boundary layer problem, including the specified boundary conditions in Eq. (4.12), is as follows

$$D\omega + M + T(h) = 0 \tag{6.1}$$

$$D=\begin{bmatrix}
K_1 & S_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
F_2 & K_2 & S_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
0 & F_3 & K_3 & S_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & & \\
A_m & 0 & \cdots & F_m & K_m & R_m & \cdots & \cdots & \cdots & 0\\
X_{m+1} & A_{m+1} & 0 & \cdots & F_{m+1} & K_{m+1} & S_{m+1} & \cdots & \cdots & 0\\
W_{m+2} & X_{m+2} & A_{m+2} & 0 & \cdots & F_{m+2} & K_{m+2} & S_{m+2} & \cdots & 0\\
0 & W_{m+3} & X_{m+3} & A_{m+3} & 0 & \cdots & F_{m+3} & K_{m+3} & S_{m+3} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & & \\
\cdots & 0 & W_{N-2} & X_{N-2} & A_{N-2} & 0 & \cdots & F_{N-2} & K_{N-2} & S_{N-2}\\
\cdots & \cdots & 0 & W_{N-1} & X_{N-1} & A_{N-1} & 0 & \cdots & F_{N-1} & K_{N-1}
\end{bmatrix}$$

and

$$M=\begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_m & r_{m+1} & r_{m+2} & \cdots & r_{N-2} & r_{N-1}\end{bmatrix}$$

where

$$r_i=\begin{cases} L_i-A_i\,w_{i-m+1}-X_i\,w_{i-m}-\widetilde{W}_i\,w_{i-m-1}-P_1\,w_0, & \text{for } 1\le i\le m-1,\\ L_i-X_i\,w_{i-m}-\widetilde{W}_i\,w_{i-m-1}, & \text{for } i=m,\\ L_i-\widetilde{W}_i\,w_{i-m-1}, & \text{for } i=m+1,\\ L_i-G_{N-1}\,w_N, & \text{for } m+2\le i\le N-1.\end{cases}$$

$$S_i=\sigma\varepsilon+\alpha h^2B_{i-1}, K_i=-2\sigma\varepsilon+2\beta h^2B_i, F_i=\sigma\varepsilon+\alpha h^2B_{i+1},$$

$$A_i=-\frac{\alpha hp_{i-1}}{2}+\frac{2\beta hp_i}{2}+\frac{3\alpha hp_{i+1}}{2}+\alpha R_{i+1}, X_i=2\alpha hp_{i-1}-2\alpha hp_{i+1}\alpha R_i,$$

$$\widetilde{W}_i=-\frac{3\alpha hp_{i-1}}{2}-\frac{2\beta hp_i}{2}+\frac{\alpha hp_{i+1}}{2}+\alpha R_{i-1}.$$

$$L_i=h^2\big[\alpha G_{i-1}+2\beta G_i+\alpha G_{i+1}\big], \qquad \text{for } i=1,2,\ldots,N-1,$$

$$T_i(h)=\mathcal{O}(h^4),$$

$$\nu=\begin{bmatrix}\nu_1\\ \nu_2\\ \vdots\\ \nu_{N-1}\end{bmatrix}, \qquad T(h)=\begin{bmatrix}T_1\\ T_2\\ \vdots\\ T_{N-1}\end{bmatrix}, \qquad O=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix},$$

which are the related vectors of Eq. (6.1).
Let

$$\tilde{\nu}=\begin{bmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \vdots\\ \tilde{\nu}_{N-1}\end{bmatrix}\simeq\nu$$

satisfy the equation

$$A\tilde{\nu}+B+T(h)=O.$$

$$A\tilde{\nu} + B = 0 \tag{6.2}$$

let

$$e_i = \nu_i - \tilde{\nu}_i, \qquad i = 1, 2, \ldots, N-1,$$

so that

$$E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \end{bmatrix} = u - U.$$

Subtracting Eq. (6.1) from Eq. (6.2), we obtain the error equation

$$DE = T(h). \tag{6.3}$$

Let $\bar{S}_i$ be the sum of the i^th row elements of the matrix A. Then we have

$$\bar{S}_i = \begin{cases} h^2\big(\alpha R_{i-1} + 2\beta R_i + \alpha R_{i+1}\big), & \text{for } 1 \le i \le m-1, \\ \dfrac{h}{2}\big(-\alpha P p_{i-1} + 2\beta P_i + 3\alpha P_{i+1}\big) + h^2\big(\alpha R_{i-1} + 2\beta R_i + \alpha R_{i+1}\big), & \text{for } i = m, \\ \dfrac{h}{2}\big(3\alpha P_{i-1} + 2\beta P_i - \alpha P_{i+1}\big) + h^2\big(\alpha R_{i-1} + 2\beta R_i + \alpha R_{i+1}\big), & \text{for } i = m+1, \\ h^2\big(\alpha R_{i-1} + 2\beta R_i + \alpha R_{i+1}\big), & \text{for } m+2 \le i \le N-1. \end{cases}$$

let

$$\zeta_{1^*} = \min_{1 \le i \le N} \big|P(t_i)\big|, \qquad \zeta_1^* = \max_{1 \le i \le N} \big|P(t_i)\big|,$$

and

$$\zeta_{2^*} = \min_{1 \le i \le N} \big|R(t_i)\big|, \qquad \zeta_2^* = \max_{1 \le i \le N} \big|R(t_i)\big|.$$

Since $0 < \varepsilon \ll 1$ and $\varepsilon \propto \mathcal{O}(h)$, with sufficiently small h, it is verified that A is irreducible and monotone [34, 36].

Therefore A^{-1} exists and $A^{-1} \ge 0$.

So, we can deduce from Eq. (6.3) that

$$\|E\| \;\le\; \|D^{-1}\|\,\|T\| \tag{6.4}$$

For small h, we have

$$\bar{S}_i \;\ge\; h^2\big[2(\alpha+\beta)\,\zeta_{2^*}\big], \qquad \begin{cases} \text{for } 1 \le i \le m-1, \\ \text{for } i = m, \\ \text{for } i = m+1, \\ \text{for } m+2 \le i \le N-1. \end{cases}$$

Let $D^{-1}_{i,k}$ be the $(i,k)^{\text{th}}$ element of D^{-1}, and we define

$$\|D^{-1}\| = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1} A^{-1}_{i,k}, \qquad \|T(h)\| = \max_{1 \le i \le N-1} \big|T_i(h)\big|.$$

$$\text{Since } D^{-1}_{i,k} \geq 0 \quad \text{and} \quad \sum_{k=1}^{N-1} D^{-1}_{i,k}\,\bar{S}_k = 1, \quad \text{for } i = 1,2,3,\ldots,N-1.$$

$$\sum_{k=1}^{m-1} D^{-1}_{i,k} \;\leq\; \frac{1}{\min_{1\leq k\leq m-1}\bar{S}_k} \;<\; \frac{1}{h^2\big[2(\alpha+\beta)\,\zeta_{2^*}\big]}, \qquad i = 1,2,3,\ldots,m-1. \tag{6.5}$$

$$D^{-1}_{i,k} \;\leq\; \frac{1}{\bar{S}_m} \;<\; \frac{1}{h^2\big[2(\alpha+\beta)\,\zeta_{2^*}\big]}, \qquad k = m,\, m+1. \tag{6.6}$$

furthere more,

$$\sum_{k=m+2}^{N-1} D^{-1}_{i,k} \;\leq\; \frac{1}{\min_{1\leq k\leq m-1}\bar{S}_k} \;<\; \frac{1}{h^2\big[2(\alpha+\beta)\,\zeta_{2^*}\big]}, \qquad i = m+2, m+3,\ldots,N-1. \tag{6.7}$$

Using Eqs. (6.5) – (6.7) and (6.4), we get

$$\|E\| \;\leq\; \mathcal{O}(h^2) \tag{6.8}$$

7. Numerical Experiments

Three problems are selected to validate the competence of the proposed scheme. The maximum absolute errors are obtained using the principle of double mesh

$$E_i^N = \max_{0\leq i\leq N} \left|\nu_i^N - \nu_{2i}^{2N}\right|$$

. The order of convergence

$$r^N = \log_2\left(\frac{E_i^N}{E_i^{2N}}\right)$$

for the examples, which are tabulated in Tables Furthermore, the calculated arithmetic solutions to the considered experiments are depicted by graphs, with and without fitting parameters for various ε and δ values.

Example 1. Consider the singularly perturbed differential–difference equation

$$\varepsilon Z''(s) + (1+s)Z'(s-\delta) + \sin(2s)\,Z(s-\delta) - e^{-s}Z(s) = \sin(2s) + 3e^{-s},$$

subject to the conditions

$$Z(s) = -1, \quad -\delta \leq s \leq 0, \qquad Z(1) = 1.$$

Example 2. Consider the singularly perturbed differential–difference equation

$$\varepsilon Z''(s) - (1+s)Z'(s-\delta) + e^{-2s}\,Z(s-\delta) - e^{-s}Z(s) = -e^{\,s-1},$$

subject to the conditions

$$Z(s) = 1, \quad -\delta \leq s \leq 0, \qquad Z(1) = -1.$$

8. Figures and Tables

Table 1: The maximum absolute errors E_i^N for Example 1 with $\delta = 0.03$

	$N=100$	$N=200$	$N=400$	$N=800$	$N=1600$
2^{-1}	8.5903×10^{-4}	3.5935×10^{-4}	1.6191×10^{-4}	7.6490×10^{-5}	3.7127×10^{-5}
2^{-2}	1.6670×10^{-3}	7.0040×10^{-4}	3.1657×10^{-4}	1.4983×10^{-4}	7.2799×10^{-5}
2^{-3}	3.1157×10^{-3}	1.3201×10^{-3}	5.9976×10^{-4}	2.8470×10^{-4}	1.3854×10^{-4}
2^{-4}	5.3960×10^{-3}	2.3196×10^{-3}	1.0636×10^{-3}	5.0739×10^{-4}	2.4757×10^{-4}
2^{-5}	8.0084×10^{-3}	3.5458×10^{-3}	1.6545×10^{-3}	7.9735×10^{-4}	3.9112×10^{-4}
2^{-6}	9.1169×10^{-3}	4.2850×10^{-3}	2.0725×10^{-3}	1.0190×10^{-3}	5.0527×10^{-4}
2^{-7}	8.4922×10^{-3}	4.3320×10^{-3}	2.1684×10^{-3}	1.0832×10^{-3}	5.4117×10^{-4}
E_i^N	1.6670×10^{-3}	1.3201×10^{-3}	1.0636×10^{-3}	1.0190×10^{-3}	1.3854×10^{-4}
R_i^N	2.3332×10^{-1}	2.1605×10^{-1}	4.2838×10^{-2}	1.9954×10^{0}	—

Table 2: The maximum absolute errors E_i^N for Example 1 with $\varepsilon = 0.01$

	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
$\delta = 0.03$	9.6947×10^{-3}	4.7126×10^{-3}	2.3224×10^{-3}	1.1533×10^{-3}	5.7480×10^{-4}
$\delta = 0.05$	3.2669×10^{-3}	1.7111×10^{-3}	8.6705×10^{-4}	4.3535×10^{-4}	2.1800×10^{-4}
$\delta = 0.06$	1.4253×10^{-3}	7.5427×10^{-4}	3.8386×10^{-4}	1.9310×10^{-4}	9.6780×10^{-5}
$\delta = 0.09$	1.0128×10^{-4}	5.4699×10^{-5}	2.8091×10^{-5}	1.4190×10^{-5}	7.1262×10^{-6}
E_i^N	1.4253×10^{-3}	1.7111×10^{-3}	2.3224×10^{-3}	1.1533×10^{-3}	2.1800×10^{-4}
R_i^N	1.8275×10^{-1}	3.0546×10^{-1}	6.9997×10^{-1}	1.6659×10^{0}	—

Table 3: The maximum absolute errors E_i^N for Example 2 with $\delta = 0.05$

	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
2^{-1}	4.8643×10^{-4}	2.1158×10^{-4}	9.7752×10^{-5}	4.6850×10^{-5}	2.2916×10^{-5}
2^{-2}	8.4229×10^{-4}	3.6752×10^{-4}	1.7011×10^{-4}	8.1613×10^{-5}	3.9942×10^{-5}
2^{-3}	1.3048×10^{-3}	5.7289×10^{-4}	2.6610×10^{-4}	1.2790×10^{-4}	6.2657×10^{-5}
2^{-4}	1.6516×10^{-3}	7.3026×10^{-4}	3.4039×10^{-4}	1.6391×10^{-4}	8.0369×10^{-5}
2^{-5}	2.5356×10^{-3}	1.3431×10^{-3}	6.9290×10^{-4}	3.5210×10^{-4}	1.7750×10^{-4}
2^{-6}	4.8419×10^{-3}	2.4668×10^{-3}	1.2456×10^{-3}	6.2601×10^{-4}	3.1383×10^{-4}
2^{-7}	4.8419×10^{-3}	2.4668×10^{-3}	1.2456×10^{-3}	6.2601×10^{-4}	3.1383×10^{-4}
E_i^N	1.3048×10^{-3}	1.3431×10^{-3}	1.2456×10^{-3}	1.2790×10^{-4}	1.7750×10^{-4}
R_i^N	2.8931×10^{-2}	7.5363×10^{-2}	2.2761×10^{0}	3.2772×10^{-1}	—

Table 4: The maximum absolute errors E_i^N for Example 5 with $\varepsilon = 0.01$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
$\delta = 0.03$	6.5162e-03	3.2475e-03	1.6315e-03	8.1941e-04	4.1086e-04
$\delta = 0.05$	2.5061e-03	1.3254e-03	6.7601e-04	3.4068e-04	1.7093e-04
$\delta = 0.06$	1.1367e-03	6.0683e-04	3.1053e-04	1.5669e-04	7.8653e-05
$\delta = 0.09$	8.4912e-05	4.6122e-05	2.3765e-05	1.2027e-05	6.0455e-06
E_i^N	1.1367e-03	1.3254e-03	1.6315e-03	1.5669e-04	1.7093e-04
R_i^N	1.5358e-01	2.0779e-01	2.3430e+00	8.6985e-02	—

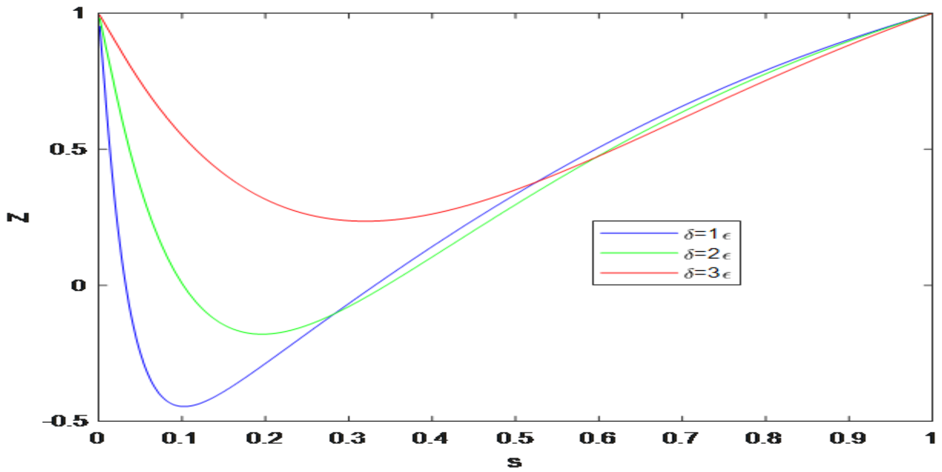

Figure 1: Solution profile for Example-1 with $\varepsilon = 0.01$ and N=800. with fitting parameter

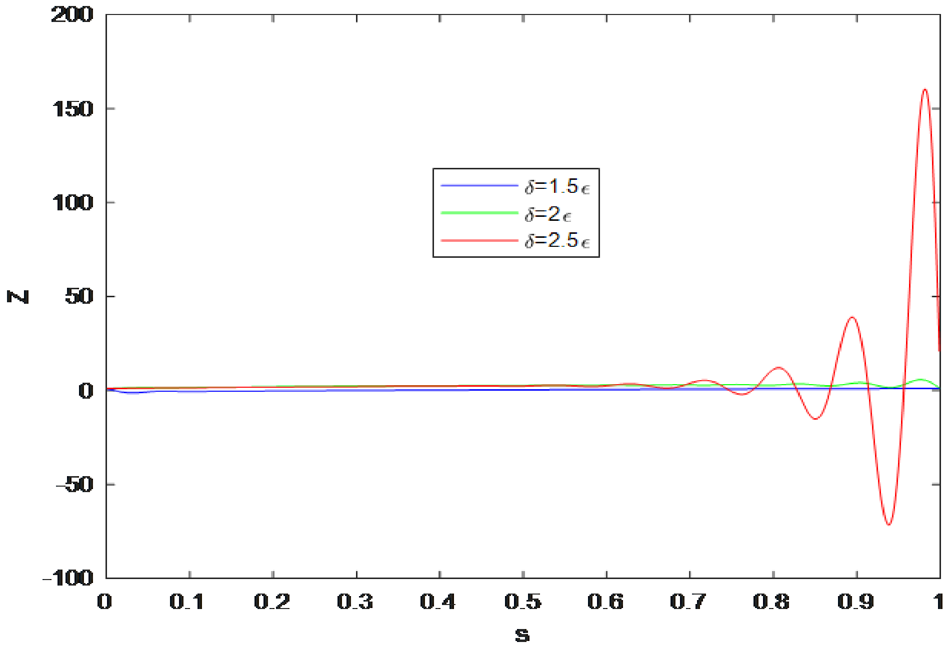

Figure 2: Solution profile for Example-1 with $\varepsilon = 0.01$ and N=800. without fitting parameter

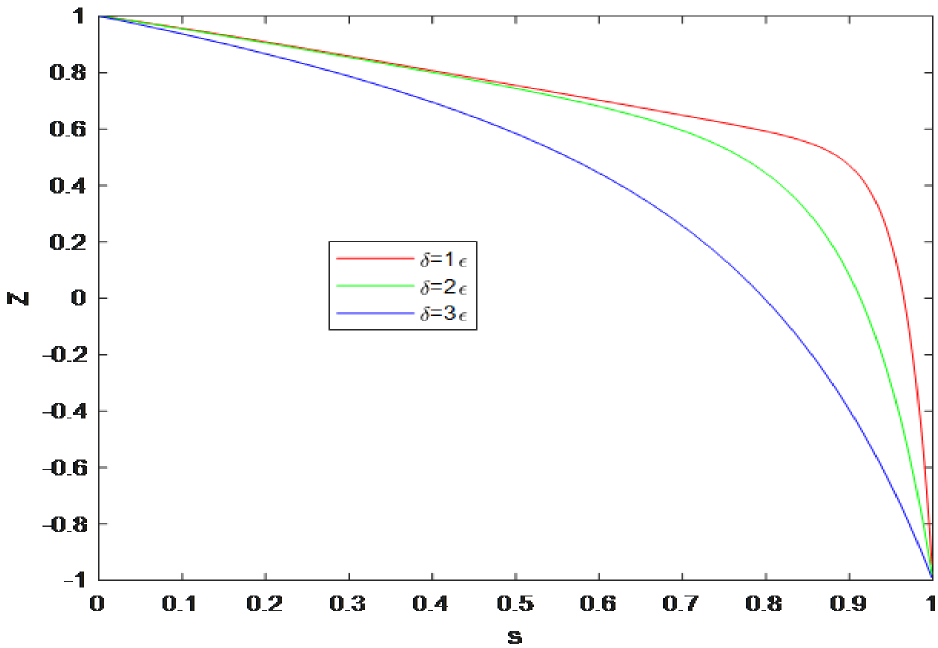

Figure 3: Solution profile for Example-2 with $\varepsilon = 0.01$ and N=800. with fitting parameter

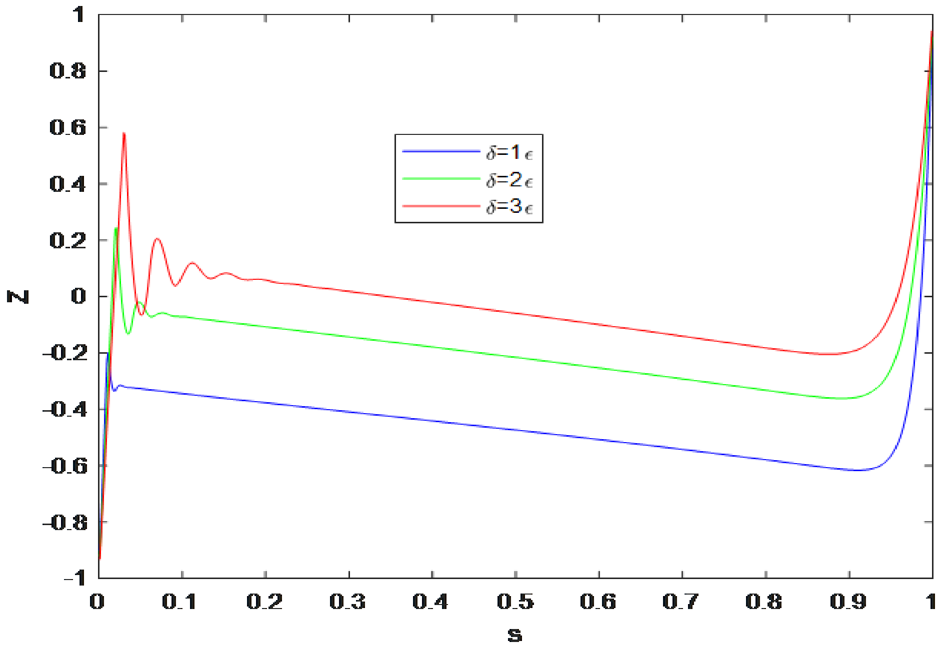

Figure 4: Solution profile for Example-2 with $\varepsilon = 0.01$ and N=800. without fitting parameter

9. Discussions and Conclusion

We consider a second-order differential–difference equation characterized by a large delay in the convection term and the presence of boundary layers at both ends of the domain. To approximate its solution, an exponential spline finite difference scheme is developed, constructed by enforcing the continuity of the first derivative at the mesh junctions. We considered the case when δ is of order $O(\varepsilon)$, layer's behaviour can change and even be destroyed, or the solution can exhibit oscillatory behaviour. To address these drawbacks in solutions, we experimented with a new scheme based on the special mesh introduced in [21], which involves adding a parameter to the exponential spline method. Table 1-4 illustrate the maximum errors as well as the order of convergence for examples 1–2. Figures depict diagrams of the solutions to the experiments for various values of δ, and we compared the graphs with the parameter for various values of $\delta = O(\varepsilon)$ to the graphs without the fitting parameter. When the δ value is larger than the $O(\varepsilon)$, the solution includes oscillations, as shown in Figures 2 and 4 (without fitting parameter), whereas in Figures 1 and 3 (with parameter), the oscillations are controlled in solutions and behaviour of the layer is well-preserved. Finally, we found that exponential spline fitting parameter method with special mesh had a significant improvement in terms of regulating the oscillations in the solutions of differential-difference equations. We have clearly noticed that the fitting parameter controls the oscillations in the layer for the large delay values. The proposed method is simple and it works very well with small delay as well as large delay.

References

1. Amala P, Lalu M and Phaneendra K, *Computational Approach to Solving a Layered Behaviour Differential Equation with Large Delay using Quadrature Scheme*, Int. J. of Applied Mechanics and Engineering. 27(4), 117-137, (2022).
2. R.Bellman, K. Cooke, *Differential-Difference Equations*. Academic Press, New York. (1963),
3. M. Bestehornand York, E.V. Grigorieva *Formation and propagation of localized states in extended systems*. Ann Phys 13: 423–431, (2004).
4. M.W. Derstine, H.M. Gibbs, F.A. Hopf, D.L. Kaplan *Bifurcation gap in a hybrid optical system*. Phys Rev A 26: 3720–3722, (1982).
5. E.P. Doolan, J.J.H. Miller, W.H.A. Schilders *Uniform Numerical Methods for Problems with Initial and Boundary Layers*. Boole Press, Dublin, (1980).
6. R.D. Driver *Ordinary and Delay Differential Equations*. Belin-Heidelberg, New York, Springer, (1977).
7. L.E. El'sgol'ts, S.B. Norkin *Introduction to the theory and application of differential equations with deviating arguments*. Academic Press, (1973).
8. V.Y. Glize *Asymptotic solution of a boundary-value problem for linear singularly perturbed functional differential equations arising in optical control theory*. J Optim Theory Appl 106: 309-335, (2000).
9. Jalil Rashidinia And Reza Jalilian *Spline solution of two-point boundary value problems*. Applied and Computational Mathematics, 9: 258-266, (2010).
10. Kadalbajoo M.K., Sharma K.K. *Numerical analysis of singularly perturbed delay differential equations with layer behavior*. Appl Math and Compu 157, 11–28, (2004).
11. Kadalbajoo M.K., Sharma K. K., *Numerical treatment of boundary value problems for second order singularly perturbed delay differential equations*. J Comput and Appl Math 24: 151–172, (2005).
12. Kadalbajoo M.K., Sharma K.K. *A Numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations*. Appl Math and Comput 197: 692-707, (2008).
13. Kadalbajoo M.K., Kumar D. *A computational method for singularly perturbed nonlinear differential-difference equations with small shift*. Applied Mathematical Modelling 34: 2584–2596, (2010).
14. P.V. Kokotovic, H.K. Khalil, J. O'Reilly *Singular perturbation methods in control analysis and design*. Academic Press, New York, (1986).
15. Y.Kuang *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, New York, (1993).
16. Kumar D. and Kadalbajoo M.K., *Numerical treatment of singularly perturbed delay differential equations using B-Spline collocation method on Shishkin mesh*. JNAIAM, 7, 73–90, (2012).
17. Lange C.G., Miura R.M. *Singular perturbation analysis of boundary-value problems for differential-difference equations. v. small shifts with layer behavior*. SIAM J Appl Math 54: 249–272, (1994).
18. Lange C.G., Miura R.M. *Singular perturbation analysis of boundary-value problems for differential-difference equations. vi. Small shifts with rapid oscillations*. SIAM J Appl Math 54: 273–283, (1994).

19. A. Lasota, M. Wazewska em Mathematical models of the red blood cell system. Mat Stos 6, 25–40, (1976).

20. Lalu M., Phaneendra Kolloju, Siva Prasad Emineni. *Numerical approach for differential-difference equations having layer behaviour with small or large delay using non-polynomial spline.* Soft Computing, 25(22), 13709-13722, (2021)

21. M.C. Mackey, L. Glass *Oscillation and chaos in physiological control systems.* Science, 197, 287–289, (1977).

22. A. Martin, S. Raun *Predetor-prey models with delay and prey harvesting.* J Math Bio 43, 247-267, (2001).

23. Mesfin Mekuria Woldaregay, Gemechis File Duressa, *Robust Numerical Scheme For Solving Singularly Perturbed Differential Equations Involving Small Delays.* Applied Mathematics E-Notes. 21, 622-633, (2021).

24. Merga Amara, Mesfin Mekuria Woldaregay and Gemechis File Duressa *Exponentially Fitted Numerical Scheme for Singularly Perturbed Differential Equations Involving Small Delays.* Journal of applied mathematics and informatics 39(3-4):419-435, (2021)

25. Mekuria Woldaregay, *Solving singularly perturbed delay differential equations via fitted mesh and exact difference Method.* Research in Mathematics, 9(1), 2109301, (2022)

26. J.J.H. Miller, R.E. O'Riordan, G.I. Shishkin. *Fitted numerical methods for singular perturbation problems.* World Scientific, Singapore, (1996).

27. R.E. O'Malley it Introduction to Singular Perturbations. Academic Press, New York, (1974).

28. Rashidinia. J, Mohammadi. R, Moatamedoshariati S.H. *Quintic Spline Methods for the Solution of Singularly Perturbed Boundary-Value Problems.* International Journal for Computational Methods in Engineering Science and Mechanics 11, 247-257, (2010).

29. Ravikanth A.S.V., Murali P. *Numerical treatment for a singularly perturbed convection delayed dominated diffusion Equation via tension splines.* Inter J of Pure and Appl Math 113, 110-118, (2017).

30. Reddy Y.N., Soujanya G.B.S.L., Phaneendra K. *Numerical Integration Method for Singularly Perturbed Delay Differential Equations.* Inter J of Appl Sci and Engin 10: 249-261, (2012).

31. Siva Prasad E., Omkar R., Phaneendra Kolloju. *Fitted parameter exponential spline method for singularly perturbed delay differential equations with a large delay.* Computational and Mathematical Methods, Hindawi 2022(1), 9291834, (2022)

32. H. Smith. *An Introduction to Delay Differential Equations with Applications to the Life Sciences.* Springer, Berlin, (2010).

33. R.B. Stein *Some models of neuronal variability.* Biophys J 7, 37-68, (1967).

34. R.S. Varga *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, New Jersey, (1962).

35. Woldaregay, M. Mekuria and Duressa, G. File. *Fitted Numerical Scheme for Singularly Perturbed Differential Equations Having Two Small Delays.* Caspian Journal of Mathematical Sciences, 11(1), 98-114. (2022)

36. D.M. Young *Iterative Solutions of Large Linear Systems.* Academic press, New York, (1971).

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