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Numerical Simulation of Multi Parameter Singularly Perturbed Two Point Boundary Value Problem

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ABSTRACT: A second-order singularly perturbed differential-difference equation involving both negative and positive shifts is examined in this paper. To obtain an approximate solution, a fitted nonpolynomial spline method is employed. The approach begins with a Taylor series expansion to derive an approximated form of the original problem, after which a fitted non-polynomial spline scheme is constructed in the form of a three-term recurrence relation. The convergence properties of the proposed method are rigorously analyzed, demonstrating a quadratic rate of convergence. Numerical experiments confirm this rate, with the maximum absolute errors reported accordingly. Additionally, the layer behaviour of the solution is investigated and illustrated through graphical representations.

Keywords: Differential - difference equation, non-polynomial spline, boundary layer.

Contents

1. Introduction

Singular perturbation problems (SPPs) arise frequently in applied science and engineering, particularly in the modelling of physical and biological systems whose solutions exhibit boundary or interior layers for certain parameter values. An SPP is characterized by the absence of a uniformly valid asymptotic expansion over the entire domain as the perturbation parameter approaches zero. The presence of narrow boundary or interior layers-regions in which the solution changes rapidly-makes both analytical and numerical treatments challenging. Within these layers, the solution varies sharply, whereas outside them it behaves smoothly and slowly. As the perturbation parameter tends to zero, the regularity of the solution deteriorates, further complicating analysis.

A singularly perturbed differential-difference equation (SPDDE) is obtained when the highest-order derivative in a differential equation is multiplied by a small positive perturbation parameter ε, and the equation additionally contains delay and/or advance arguments. These features introduce nonlocal effects that intensify the complexity of the solution structure, particularly in the presence of layers

Solving SPDDEs within boundary-constrained intervals presents significant challenges in the mathematical modelling of various physical and biological phenomena. Such equations arise in diverse applications, including initial exit-time problems in neuronal variability and activation models [18], oscillatory behaviour in the human pupil light reflex involving delayed and mixed responses [11], and analyses of red-cell system dynamics and bifurcated gaps in hybrid optical systems [2]. Comprehensive discussions on the theory and applications of SPDDEs can be found in [1], [3], [4], [13,14], and [19,20,21] .

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Several numerical techniques have been proposed to obtain accurate and reliable approximations for second-order SPDDEs with mixed (both negative and positive) shifts under boundary conditions. Asymptotic analyses of such problems were reported in [7,8,9,10]and [22]. In particular, the authors in [9] and [10] employed Taylor series expansions to study terms involving small shifts, and in [7] they examined the influence of these small shifts on the oscillatory nature of the solutions. An exponentially fitted finite difference method (FDM) for SPDDEs with delayed and advanced arguments, as well as turning-point problems, was presented in [16]. A fourth-order FDM incorporating a fitting factor was proposed in [12,25] to address similar models. Additional numerical approaches tailored for SPDDEs with mixed shifts were developed in [6]. An asymptotic expansion-based estimation technique was introduced in [17,24], while the successive complementary expansion method (SCEM) was suggested in [15,23] for solving this class of differential-difference equations.

The method presented in this study introduces a novel finite-difference approach for solving SPDDEs with mixed shifts. Section 2 provides the formal problem description and outlines the structure of the non-polynomial spline. In Section 3, a three-term difference scheme is constructed based on this spline formulation. Section 4 contains the convergence analysis of the proposed computational method. In Section 5, numerical experiments on benchmark test problems are presented to demonstrate the effectiveness of the approach, including comparisons with existing methods, followed by concluding remarks.

2. Statement of the Problem and Non-Polynomial Spline

We consider the singularly perturbed differential-difference equation with delay and advance terms

$$\varepsilon \kappa''(s)+a(s)\kappa'(s)+b(s)\kappa(s-\delta)+c(s)\kappa(s)+d(s)\kappa(s+\eta)=f(s) \tag{2.1}$$

For $s\in(0,1)$ under the boundary conditions

$$\kappa(s)=\Phi(s), -\delta\leq s\leq 0;\quad \kappa(s)=\Psi(\mathrm{s}), 1\leq \mathrm{s}\leq 1+\eta \tag{2.2}$$

Where $0<\varepsilon\ll 1$ and $a(s), b(s), c(s), d(s), t(s), \Phi(s)$ and $\Psi(\mathrm{s})$ are sufficiently smooth functions on $(0,1)$ and $0<\delta,\eta=o(\varepsilon), \delta,\eta$ are delay and advance shifts respectively

The solution of equation (2.1) with (2.2) represents the layer behaviour at each end of the interval if $a(s)-\delta b(s)+\eta d(s)>0$ then the boundary layer will be on the left side of the interval $[0,1]$ and $a(s)-\delta b(s)+\eta d(s)<0$ then the boundary layer will be on the right side of the interval $[0,1]$. Depending on the sign of $b(s)+c(s)+d(s)$, existence of boundary layers in two cases reported in [16].

The Taylor's series expansions of $\kappa(s-\delta),\kappa(s+\eta)$, we obtain

$$\kappa(s-\delta)\approx\kappa(s)-\delta\kappa'(s)+O\left(\delta^2\right) \text{ and } \kappa(s+\eta)\approx\kappa(s)+\eta\kappa'(s)+O\left(\eta^2\right) \tag{2.3}$$

Using (2.3) in (2.1), we get

$$\varepsilon\kappa''(s)+p(s)\kappa'(s)+q(s)\kappa(s)=f(s),\quad 0<s<1 \tag{2.4}$$

With the boundary conditions,

$$\kappa(s)=\Phi(0)=\Phi_0\quad \kappa(s)=\Psi(1)=\Psi_1 \tag{2.5}$$

Now we divide the interval $[0,1]$ into N equal parts with constant mesh length h . Let $s_0=0, s_1, s_2, s_3\ldots s_N=1$ be the mesh points. Then we have $s_i=ih$ and in each sub interval $[s_i,s_{i+1}], i=1,2,3,\ldots,N-1$ the non-polynomial spline is of the form

$$G_i(s)=a_i\sin\tau\left(s-s_i\right)+b_i\left(e^{-\tau(s-s_i)}+e^{\tau(s-s_i)}\right)+c_i\left(s-s_i\right)+d_i \tag{2.6}$$

Where a_i, b_i, c_i and d_i are constant coefficients and $\tau\neq 0$ arbitrary parameter.
To derive the coefficients a_i, b_i, c_i, and d_i, equation (2.6) in terms of $\kappa_i, \kappa_{i+1}, M_i$ and M_{i+1}
We define

$$G_i(s_i) = \kappa_i, \quad G_i(s_{i+1}) = \kappa_{i+1} \tag{2.7}$$
$$G_i''(s_i) = M_i, G_i''(s_{i+1}) = M_{i+1} \tag{2.8}$$

By using the conditions (2.7) and (2.8), we can get coefficients in (2.6) as

$$\begin{aligned} a_i &= \tfrac{M_i h^2}{2\varpi^2 \sin\varpi}\left(e^{-\varpi} + e^{\varpi}\right) - \tfrac{M_{i+1} h^2}{\varpi^2 \sin\varpi} \\ b_i &= \tfrac{M_i h^2}{2\varpi^2} \\ c_i &= \tfrac{\kappa_{i+1} - \kappa_i}{h} + \tfrac{M_i h^2}{h\varpi^2}\left(1 - \left(e^{-\varpi} + e^{\varpi}\right)\right) + \tfrac{M_{i+1} h^2}{h\varpi^2} \\ d_i &= \kappa_i - \tfrac{M_i h^2}{\varpi^2} \end{aligned} \tag{2.9}$$

where $\varpi = \tau h$.

Using the first derivative of continuity $M_{i-1}^{(m)}(s_i) = M_i^{(m)}(s_i)$, m $= 1$ we obtain the relation

$$\left(\kappa_{i-1} - 2\kappa_i + \kappa_{i+1}\right) = h^2\left(\alpha M_{i-1} + \beta M_i + \gamma M_{i+1}\right) \tag{2.10}$$

Where $\alpha = \frac{\varpi\left(e^{-\varpi}+e^{\varpi}\right)\cos\varpi + \left(-e^{-\varpi}+e^{\varpi}\right)\sin\varpi + 2\left(1-e^{-\varpi}-e^{\varpi}\right)\sin\varpi}{2\varpi^2\sin\varpi}$

$$\beta = \frac{-2\varpi\cos\varpi + 2\sin\varpi - \varpi\left(e^{-\varpi} + e^{\varpi}\right) - 2\left(1 - e^{-\varpi} - e^{\varpi}\right)\sin\varpi}{2\varpi^2\sin\varpi}$$
$$\gamma = \frac{\varpi - \sin\varpi}{\varpi^2\sin\varpi}$$

If $h \to 0$ then $\varpi = \tau h \to 0$ we have $(\alpha, \beta, \gamma) = \left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right)$

Equation (2.10) reduces to cubic spline [5]

3. Formulation of the Numerical Scheme

At the Node points s_i, equation (2.4) can be written as

$$\varepsilon\kappa_i'' = -p(s_i)\kappa_i' - q(s_i)\kappa_i + f(s_i) \tag{3.1}$$

Using $G_i''(s_i) = M_i = \kappa_i''$ in above equation, we get

$$\varepsilon M_j = -p_j(s_i)\kappa_i' - q_j(s_i)\kappa_i + f_j(s_i) \text{ for } j = i-1, i, i+1 \tag{3.2}$$

Substitute (3.2) in (2.10) and then using κ_j' for $j = i-1, i, i+1$

i.e

$$\kappa_i' = \frac{3\kappa_{i+1} - 2\kappa_i - \kappa_{i-1}}{4h}, \quad \kappa_{i+1}' = \frac{3\kappa_{i+1} - 4\kappa_i + \kappa_{i-1}}{2h} \text{ and } \kappa_{i-1}' = \frac{-\kappa_{i+1} + 4\kappa_i - 3\kappa_{i-1}}{2h}$$

$$\frac{\varepsilon}{h^2}\left(\kappa_{i-1} - 2\kappa_i + \kappa_{i+1}\right) = -\alpha p_{i-1}\left(\frac{-\kappa_{i+1} + 4\kappa_i - 3\kappa_{i-1}}{2h}\right) - \beta p_i\left(\frac{3\kappa_{i+1} - 2\kappa_i - \kappa_{i-1}}{4h}\right) - \gamma p_{i+1}\left(\frac{3\kappa_{i+1} - 4\kappa_i + \kappa_{i-1}}{2h}\right) - \alpha q_{i-1}\kappa_{i-1} - \beta q_i\kappa_i - \gamma q_{i+1}\kappa_{i+1} + \left(\alpha f_{i-1} + \beta f_i + \gamma f_{i+1}\right) \tag{3.3}$$

We introduce a fitting parameter $\sigma_i(\rho)$ in the above equation, we have

$$\frac{\varepsilon\sigma(\rho)}{h^2}\left(\kappa_{i-1}-2\kappa_i+\kappa_{i+1}\right)=-\alpha p_{i-1}\left(\frac{-\kappa_{\mathrm{i+1}}+4\kappa_{\mathrm{i}}-3\kappa_{\mathrm{i-1}}}{2\mathrm{h}}\right)-\beta p_i\left(\frac{3\kappa_{\mathrm{i+1}}-2\kappa_{\mathrm{i}}-\kappa_{\mathrm{i-1}}}{4\mathrm{h}}\right)-$$

$$\gamma p_{i+1}\left(\frac{3\mathrm{k}_{\mathrm{i+1}}-4\kappa_{\mathrm{i}}+\kappa_{\mathrm{i-1}}}{2\mathrm{h}}\right)-\alpha q_{i-1}\kappa_{i-1}-\beta q_i\kappa_i-\gamma q_{i+1}\kappa_{i+1} \tag{3.4}$$

$$+\left(\alpha f_{i-1}+\beta f_i+\gamma f_{i+1}\right) \tag{3.5}$$

On simplification, we obtain the following tridiagonal system

$$E_i\kappa_{i-1}+F_i\kappa_i+G_i\kappa_{i+1}=H_i,\quad i=1,2,\ldots.N-1 \tag{3.6}$$

Where

$$E_i=\varepsilon\sigma-\frac{3\alpha hp_{i-1}}{2}-\frac{\beta hp_i}{4}+\frac{\gamma hp_{i+1}}{2}+h^2\alpha q_{i-1}$$
$$F_i=-2\varepsilon\sigma+2\alpha hp_{i-1}-\frac{\beta hp_i}{2}-2\gamma hp_{i+1}+h^2\beta q_i$$
$$G_i=\varepsilon\sigma-\frac{\alpha hp_{i-1}}{2}+\frac{3\beta hp_i}{4}+\frac{3\gamma hp_{i+1}}{2}+h^2\gamma q_{i+1}$$
$$H_i=h^2\left(\alpha f_{i-1}+\beta f_i+\gamma f_{i+1}\right)$$

The above system of equation (3.5) can solve by using Thomas algorithm with the conditions

$$\kappa(0)=\Phi_0\quad\kappa(1)=\Psi_1$$

To calculate the fitting parameter, we multiply ' h ' to both side equation (3.5) and taking $h\to 0$ on equation (3.5). We use the process given in [14]. The following is an approximation for the solution of the homogeneous problem of equation (2.1).

$$\kappa(s)=\kappa_0(s)+\frac{p(0)}{p(s)}\left(\Phi(0)-\kappa_0(0)\right)e^{-\int_0^s\left(\frac{p(s)}{\varepsilon}-\frac{q(s)}{p(s)}\right)ds}+o(\varepsilon) \tag{3.7}$$

Where $\kappa_0(s)$ is the solution of $p(s)\kappa_0'(s)+q(s)\kappa_0=f_j\left(s_i\right),\kappa_0(1)=\psi_1$
By using the expansion for p(s) and q(s) about the point zero, then equation (3.6) becomes

$$\kappa(s)=\kappa_0(s)+\left(\Phi(0)-\kappa_0(0)\right)e^{-\left(\frac{p(s)}{\varepsilon}\right)s}+o(\varepsilon)$$

From (3.6), we have

$$\lim_{h\to 0}\kappa(ih)=\kappa_0(s)+\left(\Phi(0)-\kappa_0(0)\right)e^{-p(s)i\rho}$$

Using these limit values in (3.4), we obtain the following fitting factor

$$\sigma_i(\rho)=\rho\left(p_i\alpha\coth\frac{p_i\rho}{2}-\frac{\beta p_i}{4}\left(\frac{-e^{p_i\rho}-2+3e^{-p_i\rho}}{e^{p_i\rho}-2+e^{-p_i\rho}}\right)\right)\text{ where }\rho=\frac{h}{\varepsilon}$$

4. Convergence Analysis

The local error estimate for the numerical scheme of equation (3.5) is

$$T_i(h)=h^2(1-(\alpha+\beta+\gamma))\varepsilon\kappa_i''+h^3(\gamma-\alpha)\left(q_i'\kappa_i+p_i'z_i'+q_iz_i'-f_i'\right)$$
$$+h^4\frac{(\gamma+\alpha)}{2}\left(q_i''z_i+p_i''z_i'+2q_i'z_i'+2p_i'z_i''+q_iz_i''-f_i''\right)+(\gamma-\alpha)O\left(h^5\right)$$

Hence, with $(\alpha,\beta,\gamma)=\left(\frac{1}{6},\frac{4}{6},\frac{1}{6}\right)$, truncation error is of fourth order
The matrix form of equation (3.5) is

$$(D+L)\mathrm{K}+Q+T(h)=0 \tag{4.1}$$

$$\text{Where } D = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & & 0 & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix}$$

$$L = [z_i, v_i, \omega_i] = \begin{bmatrix} v_1 & \omega_1 & 0 & & 0 & 0 \\ z_2 & v_2 & \omega_2 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & z_{N-1} & v_{N-1} \end{bmatrix}$$

$$\text{Where } z_i = -\frac{3\alpha h p_{i-1}}{2} - \frac{\beta h p_i}{4} + \frac{\gamma h p_{i+1}}{2} + h^2 \alpha q_{i-1}$$

$$v_i = 2\alpha h p_{i-1} - \frac{\beta h p_i}{2} - 2\gamma h p_{i+1} + h^2 \beta q_i$$

$$\omega_i = -\frac{\alpha h p_{i-1}}{2} + \frac{3\beta h p_i}{4} + \frac{3\gamma h p_{i+1}}{2} + h^2 \gamma q_{i+1}$$

$$Q = [m_1 - (\varepsilon\sigma + z_1)\,\Phi(0), m_2, m_3 \cdots m_{N-2}, m_{N-1} - (\varepsilon\sigma + \omega_{N-1})\,\Psi(1)]^T$$

$$\text{Where } m_i = -h^2 \left(\alpha f_{i-1} + \beta f_i + \gamma f_{i+1}\right) \text{ for i=1,2,...,N-1,}$$

$$T(h) = O\left(h^4\right), \mathrm{K} = \begin{bmatrix} \mathrm{K}_1 & \mathrm{K}_2 & \cdots & \mathrm{K}_{N-1} \end{bmatrix}^T, T(h) = \begin{bmatrix} t_1 & t_2 & \cdots & t_{N-1} \end{bmatrix}^T$$

$$\text{and } O = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^T$$

Let $\kappa = \begin{bmatrix} \kappa_1 & \kappa_2 & \cdots & \kappa_{N-1} \end{bmatrix}^T \cong \kappa$ which satisfies the equation

$$(D + L)\kappa + Q = O \tag{4.2}$$

Let $e_i = \kappa_i - \mathrm{K}_i, i = 1, 2, \ldots, N-1$ denotes the discretized error so that

$$E = \begin{bmatrix} e_1 & \mathrm{e}_2 & \cdots & e_{N-1} \end{bmatrix}^T \cong \kappa - \mathrm{K}$$

Form equation (4.1) and (4.2), we obtain the error equation

$$(D + L)\mathrm{E} = T(h) \tag{4.3}$$

Let $|p_i| \le \mathcal{M}_1, |q_i| \le \mathcal{M}_2$ so that if $\mathbb{Q}_{i,j}$ so that $(i,j)^{\text{th}}$ element of matrix L , then

$$|\mathbb{Q}_{i,i+1}| = |w_i| \le \varepsilon + h(\alpha + 3\beta + 3\gamma)\mathcal{M}_1 + h^2 \alpha \mathcal{M}_2, i = 1, 2, \ldots N-2 \tag{4.4}$$

$$|\mathbb{Q}_{i,i-1}| = |z_i| \le \varepsilon + h(3\alpha + \beta + \gamma)\mathcal{M}_1 + h^2 \alpha \mathcal{M}_2, i = 2, 3, \ldots N-1 \tag{4.5}$$

As a result, for relatively small h

$$|\mathbb{Q}_{i,i+1}| < \varepsilon, i = 1, 2, \ldots N-2, |\mathbb{Q}_{i,i-1}| < \varepsilon, i = 2, 3, \ldots N-2 \tag{4.6}$$

Hence $(D + L)$ is irreducible [19]
Let the sum of element of i^{th} row of $(D + L)$ be

$$\mathbb{S}_i = -\varepsilon\sigma + \frac{3\alpha h p_{i-1}}{2} + \frac{\beta h p_i}{4} - \frac{\gamma h p_{i+1}}{2} + h^2\left(\gamma q_{i+1} + \beta q_i\right), for i = 1$$

$$S_i = h^2\left(\alpha q_{i-1} + \beta q_i + \gamma q_{i+1}\right), for i = 2, 3, \ldots, N-2$$

$$\mathbb{S}_i = -\varepsilon\sigma + \frac{\alpha h p_{i-1}}{2} - \frac{3\beta h p_i}{4} - \frac{3\gamma h p_{i+1}}{4} + h^2\left(\alpha q_{i-1} + \beta q_i\right), for i = N-1$$

Let $\mathcal{M}_{1*}=\min_{1\leq i\leq N-1}|p_i|, \mathcal{M}_1^*=\max_{1\leq i\leq N}|p_i|, \mathcal{M}_{2*}=\min_{1\leq i\leq N-1}|q_i|, \mathcal{M}_2^*=\max_{1\leq i\leq N}|q_i|$ then $0\leq\mathcal{M}_{1*}\leq\mathcal{M}_1\leq\mathcal{M}_1^*, 0\leq\mathcal{M}_{2*}\leq\mathcal{M}_2\leq\mathcal{M}_2^*$

For relatively small h , ($D+L$) is monotone [19]
Hence $(D+L)^{-1}$ exists and $(D+L)^{-1}\geq 0$.
Thus, from equation (4.3), we have

$$\|E\|\leq\|D+L\|^{-1}\|T\| \tag{4.7}$$

For relatively small h , we have Let$(D+L)_{i,j}^{-1}$ be the (i, j)$^{\text{th}}$ element of $(D+L)^{-1}$ and define

$$\|D+L\|^{-1}=\max_{1\leq i\leq N-1}\sum_{k=1}^{N-1}(D+L)_{i,k}^{-1} \text{ and } \|T(h)\|=\max_{1\leq i\leq N-1}|T_i| \tag{4.8}$$

$$\text{Since } (D+L)_{i,k}^{-1}\geq 0 \text{ and } \sum_{k=1}^{N-1}(D+L)_{i,k}^{-1}\cdot\mathbb{S}_k=1, i=1,2,\ldots N-1 \tag{4.9}$$

$$(D+L)_{i,k}^{-1}\leq\frac{1}{\mathbb{S}_i}<\frac{1}{h^2M_2}, i=1 \tag{4.10}$$

$$(D+L)_{i,k}^{-1}\leq\frac{1}{\mathbb{S}_i}<\frac{1}{h^2M_2}, i=N-1 \tag{4.11}$$

$$\sum_{k=1}^{N-1}(D+L)_{i,k}^{-1}\leq\frac{1}{\min_{2\leq i\leq N-2}\mathbb{S}_i}<\frac{1}{h^2M_2}, i=2,3,\ldots N-2 \tag{4.12}$$

With the help of three equation (4.10), (4.11), (4.12) and using equation (4.7), we have

$$\|E\|\leq O\left(h^2\right)$$

Therefore, the method is second order convergence for the scheme (3.5) with$(\alpha,\beta,\gamma)=\left(\frac{1}{6},\frac{4}{6},\frac{1}{6}\right)$

5. Numerical Results

To demonstrate the validity and robustness of the proposed method, numerical results for four test problems are presented. For each example, the maximum absolute errors (MAEs) and the corresponding rates of convergence (ROC) are computed and summarized in tabular form. Since the exact solutions of the considered problems are not available, the MAEs are estimated using the standard double-mesh technique, applying the appropriate error-estimation formula

$$E_N=\max_{0\leq i\leq N}\left|\kappa_i^N-\kappa_{2i}^{2N}\right|$$

where κ_i^N and κ_{2i}^{2N} are the computational solutions of the example problem for N and $2N$ grid points respectively. Further, the rate of convergence is determined by the formula

$$R_N=\log_2\left|\frac{E_N}{E_{2N}}\right|$$

Example 1.

$$\varepsilon\kappa''(s)+\kappa'(s)-2\kappa(s-\delta)+\kappa(s)-\kappa(s+\eta)=-1$$

with boundary conditions $\kappa(s)=1,-\delta\leq s\leq 0,\kappa(s)=1,1\leq s\leq 1+\eta$
Example 2.

$$\varepsilon\kappa''(s)+2.5\kappa'(s)-2e^s\kappa(s-\delta)-\kappa(s)-s\kappa(s+\eta)=1$$

with boundary conditions $\kappa(s)=1,-\delta\leq s\leq 0,\kappa(s)=1,1\leq s\leq 1+\eta$
Example 3.

$$\varepsilon\kappa''(s) - \left(1 + e^{-s^2}\right)\kappa'(s) - s\kappa(s-\delta) - s^2\kappa(s) - \left(1.5 - e^{-s}\right)\kappa(s+\eta) = 1$$

with boundary conditions $\kappa(s) = 1, -\delta \le s \le 0, \kappa(s) = 1, 1 \le s \le 1+\eta$

Example 4.

$$\varepsilon\kappa''(s) - \left(1 + e^{s^2}\right)\kappa'(s) - s\kappa(s-\delta) + s^2\kappa(s) - \left(1 - e^{-s}\right)\kappa(s+\eta) = 1$$

with boundary conditions $\kappa(s) = 1, -\delta \le s \le 0, \kappa(s) = -1, 1 \le s \le 1+\eta$

6. Conclusion

The numerical data in Tables 1-4 demonstrate that the proposed finite-difference method based on a non-polynomial spline yields consistently smaller maximum absolute errors (MAEs) compared with the existing methods reported in [16] and [12]. The following observations summarize the performance characteristics:

In Example 1, the proposed method shows noticeably lower errors for coarse and moderately fine meshes, while maintaining stable accuracy even for very small ε. The results in [16] display larger MAEs, particularly as ε decreases, indicating reduced stability.

Across all examples, the present method exhibits a consistent and monotonic decrease in MAE as N increases, demonstrating a reliable convergence pattern. This behaviour is clearly visible in Examples 2 and 3, where each doubling of the mesh roughly reduces the error by a factor consistent with the expected convergence rate. For very small ε $\left(10^{-3} \text{ to } 10^{-6}\right)$, the proposed method remains numerically stable, with errors stabilizing and not deteriorating. In contrast, errors in the compared methods from [16] and [12] generally increase or fluctuate, indicating reduced robustness in the boundary-layer regime. Although Example 4 is more challenging due to stronger layer effects, the proposed method still achieves lower MAEs than the method in [12], especially for finer meshes.

7. Figures and Tables

Table 1: MAE's of Example 1. With $\delta = \eta = 0.5\varepsilon$

Present method						
$\frac{\varepsilon}{h}$	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
10^{-1}	3.769×10^{-3}	9.401×10^{-4}	2.209×10^{-4}	5.439×10^{-5}	1.359×10^{-5}	3.396×10^{-6}
10^{-2}	8.914×10^{-3}	4.578×10^{-3}	2.006×10^{-3}	7.512×10^{-4}	1.893×10^{-4}	4.425×10^{-5}
10^{-3}	8.968×10^{-3}	5.034×10^{-3}	2.683×10^{-3}	1.386×10^{-3}	6.778×10^{-4}	2.835×10^{-4}
10^{-4}	8.970×10^{-3}	5.036×10^{-3}	2.684×10^{-3}	1.388×10^{-3}	7.060×10^{-4}	3.561×10^{-4}
10^{-5}	8.970×10^{-3}	5.036×10^{-3}	2.684×10^{-3}	1.388×10^{-3}	7.060×10^{-4}	3.561×10^{-4}
10^{-6}	8.970×10^{-3}	5.036×10^{-3}	2.684×10^{-3}	1.388×10^{-3}	7.060×10^{-4}	3.561×10^{-4}
Results in [16] $\delta = \eta = 0.5\varepsilon$						
10^{-1}	3.658×10^{-3}	9.595×10^{-4}	2.409×10^{-4}	6.759×10^{-5}	1.776×10^{-5}	1.232×10^{-5}
10^{-2}	1.695×10^{-2}	7.297×10^{-3}	2.486×10^{-3}	6.964×10^{-4}	1.776×10^{-4}	2.616×10^{-5}
10^{-3}	2.020×10^{-2}	1.047×10^{-2}	5.210×10^{-3}	2.461×10^{-3}	1.057×10^{-3}	3.771×10^{-4}
10^{-4}	2.052×10^{-2}	1.079×10^{-3}	5.520×10^{-3}	2.769×10^{-3}	1.363×10^{-3}	6.539×10^{-4}
10^{-5}	2.061×10^{-2}	1.088×10^{-2}	5.608×10^{-3}	2.858×10^{-3}	1.453×10^{-3}	7.417×10^{-4}
10^{-6}	1.951×10^{-2}	9.783×10^{-3}	4.513×10^{-3}	1.762×10^{-3}	3.577×10^{-4}	3.729×10^{-4}

Table 2: MAE's of Example 2. With $\delta = 0.7\varepsilon, \eta = 0.5\varepsilon$

Present Method				
$\varepsilon \downarrow N \rightarrow$	10^1	10^2	10^3	10^4
10^{-1}	1.243×10^{-2}	1.581×10^{-4}	1.585×10^{-6}	1.585×10^{-8}
10^{-2}	2.466×10^{-2}	1.819×10^{-3}	2.077×10^{-5}	2.080×10^{-7}
10^{-3}	2.448×10^{-2}	3.337×10^{-3}	1.914×10^{-4}	2.157×10^{-6}
10-4	2.490×10^{-2}	3.341×10^{-3}	3.458×10^{-4}	1.924×10^{-5}
Results in [16] $\delta = 0.7\varepsilon, \eta = 0.5\varepsilon$				
10^{-1}	1.533×10^{-2}	1.917×10^{-4}	1.921×10^{-6}	1.917×10^{-8}
10^{-2}	2.817×10^{-2}	1.865×10^{-3}	2.024×10^{-5}	2.026×10^{-7}
10^{-3}	2.853×10^{-2}	3.389×10^{-3}	1.919×10^{-4}	2.162×10^{-6}
10^{-4}	2.857×10^{-2}	3.395×10^{-3}	3.463×10^{-4}	1.925×10^{-5}

Table 3: MAE's of Example 2. With $\delta = 0.5\varepsilon, \eta = 0.5\varepsilon$

Present method						
$\varepsilon \downarrow N \rightarrow$	2^3	2^4	2^5	2^6	2^7	2^8
2^{-3}	1.132×10^{-3}	2.870×10^{-4}	7.199×10^{-5}	1.801×10^{-5}	4.504×10^{-6}	1.125×10^{-6}
2^{-4}	2.575×10^{-3}	6.692×10^{-4}	1.690×10^{-4}	4.235×10^{-5}	1.059×10^{-5}	2.649×10^{-6}
2^{-5}	5.120×10^{-3}	1.438×10^{-3}	3.712×10^{-4}	9.356×10^{-5}	2.348×10^{-5}	5.862×10^{-6}
2^{-6}	8.077×10^{-3}	2.764×10^{-3}	7.659×10^{-4}	1.969×10^{-4}	4.958×10^{-5}	1.241×10^{-5}
2^{-7}	9.475×10^{-3}	4.301×10^{-3}	1.440×10^{-3}	3.961×10^{-4}	1.016×10^{-4}	2.557×10^{-5}
2^{-8}	9.633×10^{-3}	5.026×10^{-3}	2.223×10^{-3}	7.357×10^{-4}	2.016×10^{-4}	5.166×10^{-5}
2^{-9}	9.649×10^{-3}	5.103×10^{-3}	2.592×10^{-3}	1.130×10^{-3}	3.719×10^{-4}	1.017×10^{-4}
Results in [16] $\delta = \eta = 0.5\varepsilon$						
2^{-3}	1.378×10^{-3}	3.486×10^{-4}	8.742×10^{-5}	2.187×10^{-5}	5.469×10^{-6}	1.367×10^{-6}
2^{-4}	2.880×10^{-3}	7.458×10^{-4}	1.881×10^{-4}	4.714×10^{-5}	1.179×10^{-5}	2.948×10^{-6}
2^{-5}	5.477×10^{-3}	1.526×10^{-3}	3.930×10^{-4}	9.902×10^{-5}	2.480×10^{-5}	6.204×10^{-6}
2^{-6}	8.487×10^{-3}	2.862×10^{-3}	7.898×10^{-4}	2.028×10^{-4}	5.105×10^{-5}	1.278×10^{-5}
2^{-7}	9.922×10^{-3}	4.413×10^{-3}	1.466×10^{-3}	4.024×10^{-4}	1.031×10^{-4}	2.596×10^{-5}
2^{-8}	1.009×10^{-2}	5.148×10^{-3}	2.252×10^{-3}	7.424×10^{-4}	2.032×10^{-4}	5.206×10^{-5}
2^{-9}	1.011×10^{-2}	5.228×10^{-3}	2.624×10^{-3}	1.138×10^{-3}	3.736×10^{-4}	1.021×10^{-4}

Table 4: MAE's of Example 3. With $\delta = 0.5\varepsilon, \eta = 0.5\varepsilon$

Present method						
$\varepsilon \downarrow N \rightarrow$	2^5	2^6	2^7	2^8	29	2^{10}
2^{-3}	5.102×10^{-4}	1.261×10^{-4}	3.161×10^{-5}	7.897×10^{-6}	1.973×10^{-6}	4.934×10^{-7}
2^{-4}	1.174×10^{-3}	2.837×10^{-4}	7.153×10^{-5}	1.784×10^{-5}	4.457×10^{-6}	1.114×10^{-6}
2^{-5}	2.676×10^{-3}	6.223×10^{-4}	1.550×10^{-4}	3.864×10^{-5}	9.656×10^{-6}	2.413×10^{-6}
2^{-6}	4.201×10^{-3}	1.413×10^{-3}	3.222×10^{-4}	8.159×10^{-5}	2.024×10^{-5}	5.067×10^{-6}
2^{-7}	4.818×10^{-3}	2.205×10^{-3}	7.283×10^{-4}	1.642×10^{-4}	4.195×10^{-5}	1.041×10^{-5}
2^{-8}	4.980×10^{-3}	2.520×10^{-3}	1.132×10^{-3}	3.700×10^{-4}	8.300×10^{-5}	2.129×10^{-5}
2^{-9}	4.499×10^{-3}	2.604×10^{-3}	1.290×10^{-3}	5.736×10^{-4}	1.865×10^{-4}	4.172×10^{-5}
Results in [16] $\delta = \eta = 0.5\varepsilon$						
2^{-3}	8.434×10^{-4}	2.112×10^{-4}	5.284×10^{-5}	1.321×10^{-5}	3.303×10^{-6}	8.260×10^{-7}
2^{-4}	4.172×10^{-3}	1.047×10^{-3}	2.640×10^{-4}	6.602×10^{-5}	1.650×10^{-5}	4.127×10^{-6}
2^{-5}	1.858×10^{-2}	4.743×10^{-3}	1.190×10^{-3}	2.980×10^{-4}	7.452×10^{-5}	1.864×10^{-5}
2^{-6}	6.074×10^{-2}	1.988×10^{-2}	5.080×10^{-3}	1.275×10^{-3}	3.192×10^{-4}	7.981×10^{-5}
2^{-7}	1.111×10^{-1}	6.451×10^{-2}	2.061×10^{-2}	5.270×10^{-3}	1.323×10^{-4}	3.311×10^{-4}
2^{-8}	1.297×10^{-1}	1.176×10^{-1}	6.658×10^{-2}	2.101×10^{-2}	5.372×10^{-3}	1.349×10^{-3}
2^{-9}	1.310×10^{-1}	1.372×10^{-1}	1.212×10^{-1}	6.766×10^{-2}	2.122×10^{-2}	5.425×10^{-3}

Table 5: MAE's of Example 4. With $\delta = 0.5\varepsilon, \eta = 0.5\varepsilon$

Present method					
$\varepsilon \downarrow N \rightarrow$	2^5	2^6	2^7	2^8	2^9
2^{-3}	4.896×10^{-3}	1.129×10^{-3}	2.774×10^{-4}	6.934×10^{-5}	1.731×10^{-5}
2^{-4}	1.114×10^{-2}	2.544×10^{-3}	5.859×10^{-4}	1.434×10^{-4}	3.587×10^{-5}
2^{-5}	1.898×10^{-2}	5.677×10^{-3}	1.295×10^{-3}	2.979×10^{-4}	7.289×10^{-5}
2^{-6}	2.371×10^{-2}	9.579×10^{-3}	2.864×10^{-3}	6.537×10^{-4}	1.501×10^{-4}
2^{-7}	2.448×10^{-2}	1.188×10^{-2}	4.811×10^{-3}	1.438×10^{-3}	3.282×10^{-4}
2^{-8}	2.449×10^{-2}	1.224×10^{-2}	5.950×10^{-3}	2.411×10^{-3}	7.210×10^{-4}
Results in [12] $\delta = \eta = 0.5\varepsilon$					
2^{-3}	8.354×10^{-3}	2.013×10^{-3}	4.986×10^{-4}	1.249×10^{-4}	3.121×10^{-5}
2^{-4}	$1.719 \times 10^{-}2$	4.378×10^{-3}	1.041×10^{-3}	2.571×10^{-4}	6.429×10^{-5}
2^{-5}	2.517×10^{-2}	8.889×10^{-3}	2.238×10^{-3}	5.290×10^{-4}	1.303×10^{-4}
2^{-6}	3.154×10^{-2}	1.294×10^{-2}	4.516×10^{-3}	1.131×10^{-3}	2.664×10^{-4}
2^{-7}	4.478×10^{-2}	1.622×10^{-2}	6.559×10^{-3}	2.276×10^{-3}	5.686×10^{-4}
2^{-8}	7.878×10^{-2}	2.317×10^{-2}	8.224×10^{-3}	3.301×10^{-3}	1.142×10^{-3}

Table 6: MAE's of Example 4. With $\varepsilon = 0.1$

$N \rightarrow$	10^1	10^2	10^3	10^4
Present Method				
$\delta \downarrow$	$\eta = 0.5\varepsilon$			
0.00	5.873×10^{-2}	5.918×10^{-4}	5.831×10^{-6}	5.829×10^{-8}
0.05	5.823×10^{-2}	5.841×10^{-4}	5.753×10^{-6}	5.752×10^{-8}
0.09	5.783×10^{-2}	5.779×10^{-4}	5.569×10^{-6}	5.682×10^{-8}
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	5.806×10^{-2}	5.804×10^{-4}	5.714×10^{-6}	5.711×10^{-8}
0.05	5.823×10^{-2}	5.841×10^{-4}	5.753×10^{-6}	5.752×10^{-8}
0.09	5.837×10^{-2}	5.871×10^{-4}	5.783×10^{-6}	5.782×10^{-8}
Results in [12]				
$\delta \downarrow$	$\eta = 0.5\varepsilon$			
0.00	9.109×10^{-2}	1.112×10^{-2}	6.382×10^{-4}	4.004×10^{-5}
0.05	9.047×10^{-2}	1.095×10^{-2}	6.306×10^{-4}	3.950×10^{-5}
0.09	8.996×10^{-2}	1.082×10^{-2}	6.244×10^{-4}	3.903×10^{-5}
$\eta \downarrow$	$\delta = 0.5\varepsilon$			
0.00	9.604×10^{-2}	1.116×10^{-2}	6.458×10^{-4}	3.924×10^{-5}
0.05	9.621×10^{-2}	1.124×10^{-2}	6.494×10^{-4}	3.950×10^{-5}
0.09	9.634×10^{-2}	1.131×10^{-2}	6.522×10^{-4}	3.970×10^{-5}

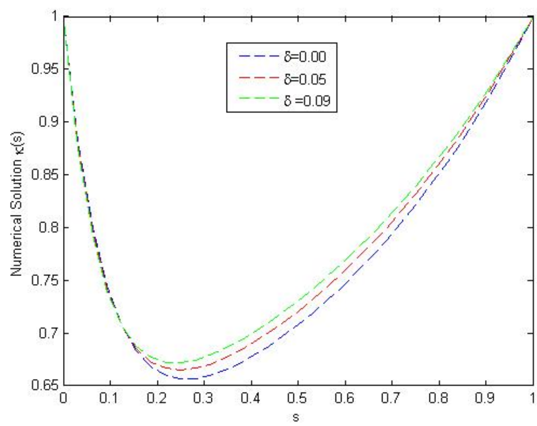

Figure 1: Layer behaviour of Example 1 with $\delta, N = 2^5, \varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

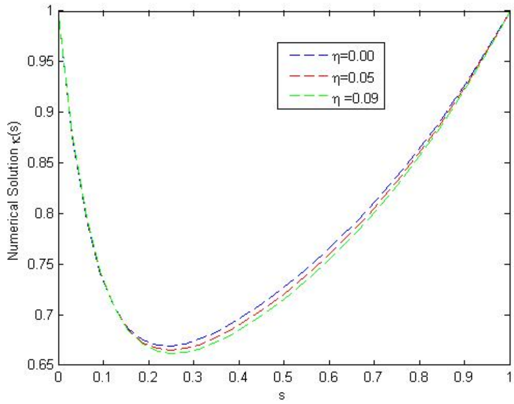

Figure 2: Layer behaviour of Example 1 with $\eta, N = 2^5, \varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

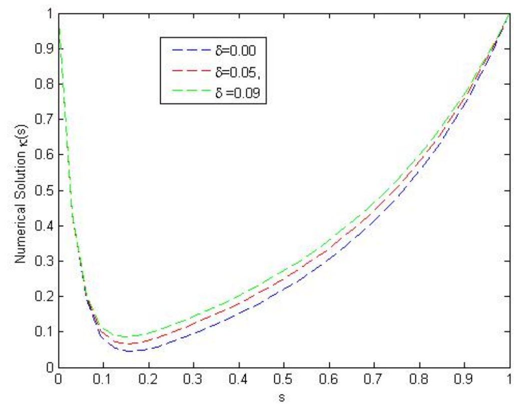

Figure 3: Layer behaviour of Example 2 with $\delta, N = 2^5, \varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

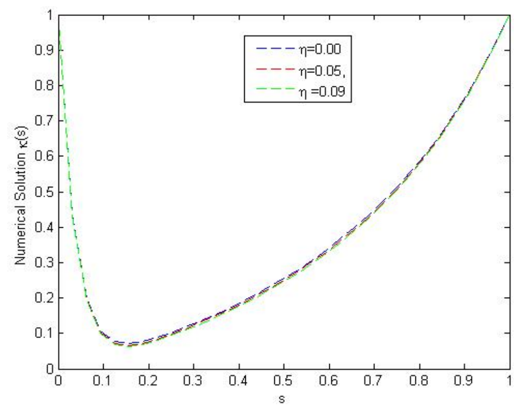

Figure 4: Layer behaviour of Example 2 with $\eta, N = 2^5, \varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

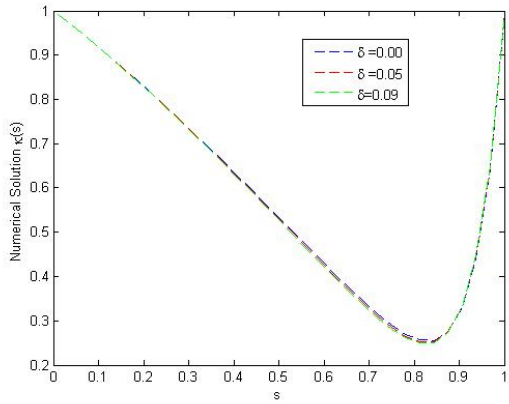

Figure 5: Layer behaviour in Example 3 with $\delta, N = 2^5, \varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

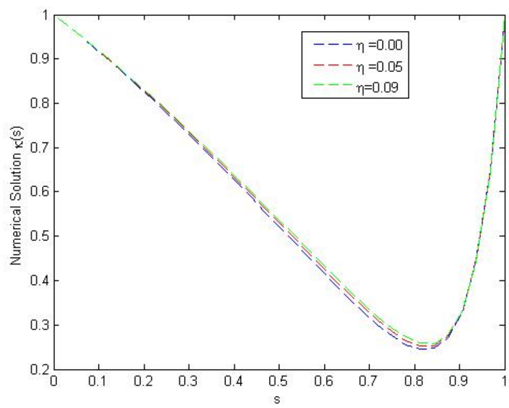

Figure 6: Layer behaviour of Example3 with $\eta, N = 2^5, \varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

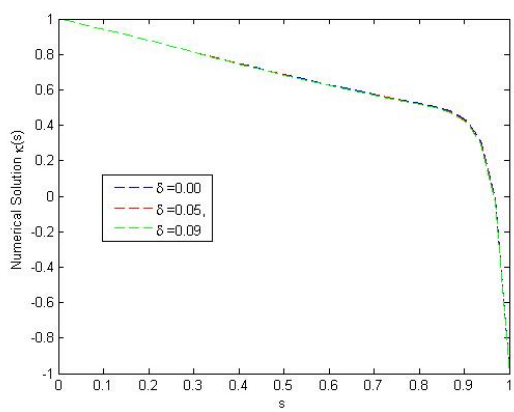

Figure 7: Layer behaviour in Example 4 with $\delta, N = 2^5, \varepsilon = 10^{-1}$ and $\eta = 0.5\varepsilon$

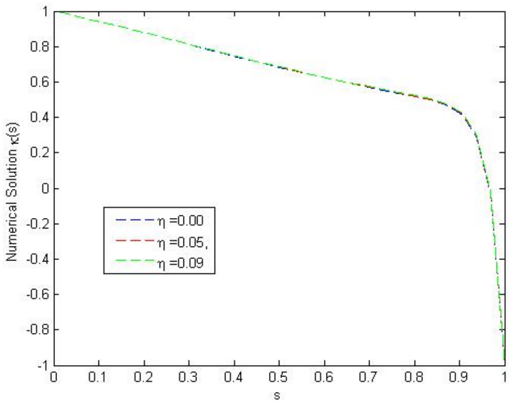

Figure 8: Layer behaviour of Example 4 with $\eta, N = 2^5, \varepsilon = 10^{-1}$ and $\delta = 0.5\varepsilon$

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