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Classification of Control Operators for a Class of Distributed Delayed Systems with Application of Diffusion Case

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ABSTRACT: This work investigates the classification of spatio-temporal (distributed parameter) systems, with a focus on the concept of domination in a class of delayed distributed systems. We examine the possibility of classifying such systems based on the structure of input and output operators, even when the underlying dynamics differ. Precise definitions of weak and exact domination are proposed, along with key properties and characterization results for general controlled systems and associated input operators. The analysis extends to the case of multi-actuator and multi-sensor configurations.Applications to parabolic systems are provided to illustrate the theoretical results. Furthermore, the relationship between domination and the problem of exact reachability with minimum energy is explored.

Keywords: Dynamical systems, control, delays, distributed systems, domination.

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1. Introduction

Delay systems are very important for understanding how real-world processes change over time. They reflect the inherent hereditary nature of many systems by incorporating past information to predict future evolution. In practical scenarios, the effect of a control input $w(t)$ at time t does not instantaneously influence the state $x(t)$, rather the evolution of the state often depends on a previous input $w(t-h)$, where h denotes the delay intrinsic to the system or control mechanism.

The nature and magnitude of this delay h can vary depending on the characteristics of the system and the control applied. The examination of dynamical systems with delays has recently garnered considerable interest owing to its applicability across several scientific and engineering fields. Delays appear in various forms including constant, time-varying, and multiple delays, each introducing distinct analytical and control challenges (see [20], [21], [22], [9]). These arise naturally in fields such as population dynamics, epidemiology, neural networks, and feedback control systems, where the influence of past states cannot be neglected. Recent advances have extended classical frameworks to incorporate time-varying and impulsive delays [19], fractional-order systems with delayed feedback ([14], [24]), and hybrid control designs with fixed-time stability guarantees [12]. This growing body of literature highlights the centrality of delay analysis in modern applied mathematics and control theory.

In control theory, it is well established that certain control configurations can outperform others. This insight leads to the classification of input operators, giving rise to the notion of *domination*. Initially

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developed in the context of parabolic systems and later extended to hyperbolic systems (see [1,5,7]), the concept of domination provides a framework for comparing and classifying actuators (input operators) and, via duality, understanding sensors (output operators).

Importantly, the number of actuators alone does not determine domination. In some cases, a single actuator may dominate several others. This framework extends to asymptotic settings and regional analyses (see [1,6]). In prior work [18], we studied domination in the fractional-order setting for linear disturbed systems with finite-dimensional states, connecting it to the concept of *remediability* [8].

A further dimension of the theory involves comparing systems $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ with identical dynamics $\mathcal{A}$ but distinct control operators D_1 and D_2. In this article, we extend this idea to delayed distributed systems $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ governed by potentially same dynamic $\mathcal{A}$. Our goal is to investigate the problem of domination and classification in such systems with regard to an output operator Q, including the delayed versions of diffusion processes, actuators, and sensors.

The concept is further extended to observed systems through a duality-based framework. We characterize fundamental properties of domination and explore connections with related concepts in optimal control and exact reachability.

The further structure of this paper is as follows.

In Section 2, we present a generalized delayed distributed system and recast it into an equivalent state-space formulation. We also introduce the concepts of exact dominance and weak dominance, highlighting their main properties. In section 3, we apply it to a parabolic system and look at the normal case of sensors and actuators. In the last section 4, we present other important applications and properties concerning practical and applied sides of the concept of domination. Indeed, we studied the relationship between this concept and some useful optimal control and rechability problems.

2. Domination problem

In this part, we examine a generic category of dynamical systems characterized by delays in the control, as delineated by the subsequent equations

$$(\mathcal{K}_1)\begin{cases} \dot{x_1}(t) = \mathcal{A}x_1(t) + D_1 w_1(t-\tau_1) & 0 < t < T \\ x_1(0) = x_0 & \\ w_1(\beta) = 0 \quad \beta \in [-\tau_1, 0] & \tau \geq 0 \end{cases} \tag{2.1}$$

$$(\mathcal{K}_2)\begin{cases} \dot{x_2}(t) = \mathcal{A}x_2(t) + D_2 w_2(t-\tau_2) & 0 < t < T \\ x_2(s) = x_0 & \\ w_2(\beta) = 0 \quad \beta \in [-\tau_2, 0] & \tau \geq 0. \end{cases} \tag{2.2}$$

The operator $\mathcal{A}$ is assumed to be a second-order differential operator with compact resolvent, which generates a $(\mathbb{S}(t))_{t\geq 0}$-semigroup on the Hilbert space X (the state space), $D_i \in \mathcal{L}(U, X), i = 1, 2$ are input operators defined on the control space U (also a Hilbert space). The constants τ_1, τ_2 represent delays such that $0 \leq \tau_1 < \tau_2$, the delayed inputs w_i are presumed to reside in the extended state spaces $L^2(-\tau_i, T; W)$ including the system's history. The systems (2.1) and (2.2) are augmented with the output equations, respectively:

$$(\mathcal{E}_i) \quad y_i(t) = Qx_i(t)$$

$Q \in \mathcal{L}(X, Y)$ is the output operator, Y is a Hilbert space.

To resolve $\mathcal{K}_1$ and $\mathcal{K}_2$, we employ the semigroup approach, a method commonly utilized in control theory and well-suited for numerical simulations (see [10]). Conversely, considering the zero initial control for $t \in [-\tau_i, 0]$, hereafter we define a corresponding semigroup of left shift (see [23]):

$$\Psi_i(t)w_i = \begin{cases} w_i(t-\tau_i), & t-\tau_i \geq 0 \\ 0, & t-\tau_i \leq 0. \end{cases}$$

The adjoint is defined as the right shift semigroup as follows

$$\Psi_i^*(t)w_i = \begin{cases} w_i\,(t+\tau_i)\,, & t+\tau_i \geq 0 \\ 0, & t+\tau_i \leq 0. \end{cases}$$

It is important to recognize that Ψ and Ψ^* may also be defined for $t \in \mathbb{R}$ [23].
We subsequently describe the delayed inputs ([15]- [17]) and the model state solution of the systems $\mathcal{K}_i$ as follows:

$$x_i(t) = \mathbb{S}(t)x_0 + \int_0^t \mathbb{S}(t-s)D_i\Psi_i(s)w_i ds.$$

At the final time T, where T is greater than τ_i, the observation is as follows

$$y_i(T) = Q\mathbb{S}(T)x_0 + \int_0^T Q\mathbb{S}(T-s)D_i\Psi_i(s)w_i ds.$$

We consider the operators

$$\begin{array}{rcl} H_1 : L^2(0,T;W_1) & \rightarrow & X \\ w_1 & \rightarrow & \int_0^T \mathbb{S}(t-s)D_1\Psi_1(t)w_1 ds \end{array}$$

and

$$\begin{array}{rcl} H_2 : L^2(0,T;W_2) & \rightarrow & X \\ w_2 & \rightarrow & \int_0^T \mathbb{S}(t-s)D_2\Psi_2(t)w_2 ds \end{array}$$

the output is expressed as follows

$$y_i(t) = Q\mathbb{S}(t)x_0 + QH_iw_i. \tag{2.3}$$

This paper presents the concept of domination in the context of delayed and disturbed systems. The objective is to examine a potential comparison between the input operators D_1 and D_2 (or systems $\mathcal{K}_1$ and $\mathcal{K}_2$ if the systems have different dynamics) in relation to the output operator Q. It is based on the dynamic $\mathcal{A}$, the control operators D_1, D_2 and the observation operator Q. We subsequently present the relevant concept of domination.
This leads to the following definitions.

Definition 2.1 -If

$$Im(QH_2) \subset Im(QH_1).$$

than $\mathcal{K}_1$ dominates exactly $\mathcal{K}_2$, with respect to Q on $[0,T]$.
-$\mathcal{K}_1$ dominates weakly $\mathcal{K}_2$, with respect to Q on $[0,T]$, if

$$\overline{Im(QH_2)} \subset \overline{Im(QH_1)}.$$

In this context, we note respectively

$$(\mathcal{A},\mathcal{K}_2) \underset{Q}{\leq} (A,\mathcal{K}_1) \text{ and } (\mathcal{A},\mathcal{K}_2) \underset{Q}{\lesssim} (\mathcal{A},\mathcal{K}_1)\,.$$

In addition, the following properties and observations are presented.

Remark 2.1 *1. Exact domination concerning an output operator Q clearly implies weak domination. The opposite is not true.*

2. If the system $(\mathcal{K}_1)$ is exactly (respectively weakly) controllable, which is equivalent to

$$Im(H_{D_1}) = X(\text{ respectively } , \overline{Im(H_{D_2})} = X)$$

then the system $(\mathcal{K}_1)$ exactly (or weakly) dominates any system $(\mathcal{K}_2)$, concerning to any output operator Q.

3. In linear systems, the concept of domination is independent of state delay and the initial state $x_i(0)$.

In the following, we provide characterization results for exact and weak domination.

Proposition 2.1 *The subsequent properties are equivalent:*
1- $\mathcal{K}_1$ dominates exactly $\mathcal{K}_2$, with respect to Q on $[0,T]$.
2- $\forall w_2 \in L^2(0,T;W_2)$, $\exists\, w_1 \in L^2(0,T;W_1)$ satisfying

$$QH_1w_1 + QH_2w_2 = 0.$$

3- $\exists\, \gamma > 0$ such that, $\forall\, \xi \in \mathrm{Y}$, the following inequality holds:

$$\left\|\Psi_2^*(.)D_2^*\mathbb{S}^*(T-.)Q^*\xi\right\|_{L^2(0,T;W_2)} \le \gamma \left\|\Psi_1^*(.)D_1^*\mathbb{S}^*(T-.)Q^*\xi\right\|_{L^2(0,T;W_1)}.$$

Proof: The equivalence between 1) and 2) comes from the definition. Concerning the equivalence $2 \Leftrightarrow 3$ follows from the subsequent result if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces, and $\mathcal{P} \in \mathcal{L}(\mathcal{X},\mathcal{Z})$, $Q \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, then

$$I(\mathcal{P}) \subset I(\mathcal{Q})$$

if and only if

$$\exists \beta > 0,\ \forall z^* \in \mathcal{Z}' \ /\ \left\|\mathcal{P}^*z^*\right\|_{\mathcal{X}'} \le \beta \left\|\mathcal{Q}^*z^*\right\|_{\mathcal{Y}'},$$

where $\mathcal{X}'$, $\mathcal{Y}'$ and $\mathcal{Z}'$ represent the dual spaces of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. □

The weak domination arises as a consequence of the following general result.

Proposition 2.2 *$\mathcal{K}_1$ dominates weakly $\mathcal{K}_2$, on $[0,T]$ with respect to Q, if and only if*

$$Ker[(\Psi_1^*D_1^*\mathbb{S}(.)^*Q^*)] \subset Ker[(\Psi_2^*D_2^*\mathbb{S}(.)^*Q^*)].$$

Proof: The conclusion is a direct consequence of the definition and of the fact that the orthogonal complement inclusion

$$\overline{Im(QH_2)}^{\perp} \subset \overline{Im(QH_1)}^{\perp}$$

can be equivalently rewritten as

$$Ker[(QH_1)^*] \subset Ker[(QH_2)^*],$$

and from

$$(QH_i)^* = \Psi_i^*D_i^*\mathbb{S}^*Q^*.$$

□

3. Application to diffusion systems

This section examines a class of parabolic systems, specifically addressing the concepts of actuators and sensors [4,2,3], understood as input and output operators. In what follows, we work in the Hilbert space $X = L^2(\mathcal{O})$ without loss of generality and adopt an analytical setting in which the operator $\mathcal{A}$ is assumed to generate a strongly continuous semigroup $(\mathbb{S}(T))_{t\ge 0}$ defined by

$$\mathbb{S}(T)x = \sum_{n=1}^{+\infty} e^{\mu_n t} \sum_{j=1}^{r_n} \langle x, \eta_{nj}\rangle\, \eta_{nj} \tag{3.1}$$

Let $\{\eta_n^j\ ;\ j = 1,\dots,r_n,\ n \ge 1\}$ denote a complete orthonormal family of eigenfunctions of the operator $\mathcal{A}$, associated with a sequence of real eigenvalues $(\mu_n)_{n\ge1}$ satisfying

$$\mu_1 > \mu_2 > \mu_3 > \cdots.$$

Here, r_n represents the algebraic multiplicity of the eigenvalue μ_n. Under these assumptions, the operator A is defined by

$$\mathcal{A}x = \sum_{n=1}^{+\infty} \mu_n \sum_{j=1}^{r_n} \langle x, \eta_{nj} \rangle \eta_{nj}. \tag{3.2}$$

When the system $(\mathcal{K}_1)$ is driven by p zone actuators $(\mathcal{O}_i, l_i)_{1\leq i\leq p}$, it follows that $W_1 = \mathbb{R}^p$ and

$$D_1 w_{t-\tau_1} = \sum_{i=1}^{p} l_i w_{t-\tau_1,i} \tag{3.3}$$

where $w_{t-\tau_1} = (w_{t-\tau_1,1}, \ldots, w_{t-\tau_1,p})^{tr} \in L^2(-\tau_1, 0; \mathbb{R}^p)$ and $l_i \in L^2(\mathcal{O})$; $\mathcal{O}_i = supp(l_i) \subset \mathcal{O}$. The adjoint of the control operator is then given by

$$D_1^* x = (\langle l_1, x\rangle, ..., \langle l_p, x\rangle)^{tr}. \tag{3.4}$$

Similarly, when $(\mathcal{K}_2)$ is actuated by q zone actuators $(\Omega_i, k_i)_{1\leq i\leq q}$, the corresponding control space is $W_2 = \mathbb{R}^q$ and the adjoint operator takes the form

$$D_2 v_{t-\tau_2} = \sum_{i=1}^{q} k_i v_{t-\tau_2,i} \tag{3.5}$$

with $v_{t-\tau_2} = (v_{t-\tau_2,1}, ..., v_{t-\tau_2,p})^{tr} \in L^2(-\tau_2, 0; \mathbb{R}^q)$ and $k_i \in L^2(\mathcal{O})$; $\Omega_i = supp(k_i) \subset \mathcal{O}$. we have

$$D_2^* x = (\langle k_1, x\rangle, ..., \langle k_q, x\rangle)^{tr}. \tag{3.6}$$

As shown in the following section, these representations yield characterization results that depend on the operator Q and the associated controllability matrix, rather than on the observability matrix, in the case where the observation is performed through a finite number of sensors. Initially, we will demonstrate the subsequent preliminary result.

Proposition 3.1 *We have*

$$Ker(\Psi_1^* D_1^* \mathbb{S}^*(.)Q^*) = \{\xi \in Y^{'} / \forall n \in \mathbf{N}^*, (\langle Q^*\xi, \eta_{nj}\rangle)_{1\leq j\leq r_n} \in Ker(N_n)\}$$

and

$$Ker(\Psi_2^* D_2^* \mathbb{S}^*(.)Q^*) = \{\xi \in Y^{'} / \forall n \in \mathbf{N}^*, (\langle Q^*\xi, \eta_{nj}\rangle)_{1\leq j\leq r_n} \in Ker(R_n)\}$$

where N_n and R_n denote the respective controllability matrices defined by:

$$N_n = (\langle l_i, \eta_{nj}\rangle)_{1\leq i\leq p; 1\leq j\leq r_n} \text{ and } R_n = (\langle k_i, \eta_{nj}\rangle)_{1\leq i\leq q; 1\leq j\leq r_n}. \tag{3.7}$$

Proof: We possess

$$\Psi_1^* D_1^* \mathbb{S}^*(T-.)Q^*\xi = \left(\sum_{n=1}^{+\infty} e^{\mu_n (T-\tau_1-.)} \sum_{j=1}^{r_n} \langle Q^*\xi, \eta_{nj}\rangle \langle l_i, \eta_{nj}\rangle \right)_{1\leqslant i\leqslant p}.$$

Therefore, $\xi \in ker(\Psi_1^* D_1^* \mathbb{S}^*(.)Q^*)$ if and only if

$$\sum_{n=1}^{+\infty} e^{\mu_n (t-\tau_1-.)} \sum_{j=1}^{r_n} \langle Q^*\xi, \eta_{nj}\rangle \langle l_i, \eta_{nj}\rangle = 0; \forall i \in 1, ..., p, \forall t + \tau_1 \geq 0.$$

According to analyticity, this is equivalent to

$$\sum_{j=1}^{r_n} \langle Q^*\xi, \eta_{nj}\rangle \langle l_i, \eta_{nj}\rangle = 0; \forall n \geq 1, \forall i \in 1, ..., p$$

or

$$\Psi_1^* D_1^* \mathbb{S}^*(.) Q^* \xi = 0 \Longleftrightarrow v_n(\xi) \in \ker\left(N_n\right), \forall n \geq 1$$

or

$$v_n(\xi) = (\langle Q^*\xi, \eta_{nj}\rangle)_{j=1,r_n}.$$

The demonstration of the second proposition is analogous. □

The subsequent result, obtained from Proposition(3.1), provides characterizations of exact and weak domination concerning actuators.

Proposition 3.2 ***i.*** *($\mathcal{K}1$) dominates exactly the system ($\mathcal{K}2$) with respect to the operator Q, if and only if there exists $\gamma > 0$ such that for all $\xi \in Y'$, we have*

$$\begin{aligned} & \left\| \left(\sum_{n=1}^{+\infty} e^{\mu_n(t-\tau_2-.)} \sum_{j=1}^{r_n} \langle Q^*\xi, \eta_{nj}\rangle \langle k_i, \eta_{nj}\rangle \right)_{1\leqslant i\leqslant q} \right\|_{L^2(0,T;\mathbb{R}^q)} \\ \leq \gamma & \left\| \left(\sum_{n=1}^{+\infty} e^{\mu_n(t-\tau_1-.)} \sum_{j=1}^{r_n} \langle Q^*\xi, \eta_{nj}\rangle \langle l_i, \eta_{nj}\rangle \right)_{1\leqslant i\leqslant p} \right\|_{L^2(0,T;\mathbb{R}^p)} . \end{aligned} \tag{3.8}$$

ii. *($\mathcal{K}_1$) dominates weakly the system ($\mathcal{K}_2$) with respect to the operator Q, if and only if for all $\xi \in Y'$, we have*

$$\left[\forall n \in \mathbb{N}^*, \left(\langle Q^*\xi, \eta_{nj}\rangle\right)_{1\leqslant j\leqslant r_n} \in \ker N_n\right] \Rightarrow \left[\forall n \in \mathbb{N}^*, \left(\langle Q^*\xi, \eta_{nj}\rangle\right)_{1\leqslant j\leqslant r_n} \in \ker R_n\right]. \tag{3.9}$$

This leads to the following definition.

Definition 3.1 If ($\mathcal{K}_1$) exactly (or weakly) dominates ($\mathcal{K}_2$) with respect to Q, we say that the actuator $(\mathcal{O}_i, l_i)_{1\leq i\leq p}$ exactly (or weakly) dominates $(\Omega_i, k_i)_{1\leq i\leq q}$ with respect to Q.

Typically, observation is facilitated by sensors. The subsequent section analyzes this case. Let the output be represented by m sensors $(E_i, e_i)_{1\leq i\leq m}$, we have

$$Qz = \begin{pmatrix} \langle z, e_1\rangle \\ \vdots \\ \langle z, e_m\rangle \end{pmatrix} \in \mathbb{R}^m$$

and

$$Q^*\xi = \sum_{i=1}^{m} \xi_i e_i \text{ for } \xi \in \mathbb{R}^m.$$

We have the following proposition

Proposition 3.3 *($\mathcal{K}_1$) dominates weakly the system ($\mathcal{K}_2$) with respect to the sensors $(E_i, e_i)_{1\leq i\leq m}$, if and only if*

$$\bigcap_{n\geq 1} \ker\left(N_n J_n^{tr}\right) \subset \bigcap_{n\geq 1} \ker\left(R_n J_n^{tr}\right) \tag{3.10}$$

where J_n represent the respective observability matrice defined as follow

$$J_n = (\langle e_i, \eta_{nj}\rangle)_{1\leq i\leq m; 1\leq j\leq r_n}. \tag{3.11}$$

Proof:
By definition $(\mathcal{K}_1)$ dominates the system $(\mathcal{K}_2)$ weakly with respect to the sensors $(E_i, e_i)_{1\leq i\leq m}$, if and only if, for all $\xi = (\xi_k)_{1\leq k\leq m} \in \mathbb{R}^m$,

$$\left(\sum_{i=1}^{m} \xi_k \left\langle e_k, \eta_{nj}\right\rangle\right)_{1\leq j\leq r_n} \in \ker\left(N_n\right), \forall n \in \mathbb{N}^*$$

implies that

$$\left(\sum_{i=1}^{m} \xi_k \left\langle e_k, \eta_{nj}\right\rangle\right)_{1\leq j\leq r_n} \in \ker\left(R_n\right), \forall n \in \mathbb{N}^*$$

or equivalently, for any $\xi \in \mathbb{R}^m$,

$$[\xi \in \ker(N_n J_n^*), \forall n \geq 1] \Rightarrow [\xi \in \ker(R_n J_n^*), \forall n \geq 1] \tag{3.12}$$

we then have the result.

□

We will provide the following observations

Remark 3.1 *1. A single actuator may dominates several actuators* ($p > 1$)*, with respect to an output operator* Q *(sensors).*

2. For a system comprising one actuator and one sensor, i.e. for $p = q = 1$ *and* $m = 1$*, we have*

$$N_n = \left(\left\langle l, \eta_{n1}\right\rangle, \cdots, \left\langle l, \eta_{nr_n}\right\rangle\right), R_n = \left(\left\langle k, \eta_{n1}\right\rangle, \cdots, \left\langle k, \eta_{nr_n}\right\rangle\right)$$

and

$$J_n^{tr} = \begin{pmatrix} \left\langle e, \eta_{n1}\right\rangle \\ \vdots \\ \left\langle e, \eta_{nr_n}\right\rangle \end{pmatrix}.$$

Then

$$\begin{aligned} N_n J_n^{tr} &= \left(\sum_{j=1}^{r_n} \left\langle l, \eta_{nj}\right\rangle \left\langle e, \eta_{nj}\right\rangle\right), \\ R_n J_n^{tr} &= \left(\sum_{j=1}^{r_n} \left\langle k, \eta_{nj}\right\rangle \left\langle e, \eta_{nj}\right\rangle\right). \end{aligned} \tag{3.13}$$

3. For a finite number of sensors, exact domination and weak domination are equivalent.

3.1. Applications

To illustrate prior results, we consider a class of diffusion systems characterized by the following parabolic equation

$$(\mathcal{S}) \begin{cases} \frac{\partial z(x,t)}{\partial t} & = \Delta z(x,t) + l(x) w(t-\tau) & \mathcal{O}\times]0,T[\\ z(x,0) & = 0 & \mathcal{O} \\ z(\xi,t) & = 0 & \partial\mathcal{O}\times]0,T[\\ w(\beta) & = 0 & \beta \in [-\tau, 0] \end{cases} \tag{3.14}$$

where $\mathcal{O}$ is a bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary $\partial\mathcal{O} = \Gamma$; $Z = L^2(\mathcal{O})$ and $\mathcal{A}z = \Delta z$ for $z \in D(\mathcal{A}) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. $\mathcal{K}$ is enhanced by the output equation

$$(\mathcal{E})\; y(t) = Cz(t), 0 < t < T. \tag{3.15}$$

This study analyzes the case of one-dimensional space.

3.2. One Dimension Case

In this section, we examine the systems $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ characterized by the following one-dimensional equations, with $\mathcal{O} =]0, a[$ et $\mathcal{A} = \Delta$.

$$(\mathcal{K}_1) \begin{cases} \frac{\partial z_1}{\partial t}(x,t) & = \frac{\partial^2 z_1}{\partial x^2}(x,t) + l(x)w_1(t-\tau_1) &]0,a[\times]0,T[\\ z_1(0,t) & = z_1(a,t) = 0 &]0,T[\\ z_1(x,0) & = 0 &]0,a[\\ w_1(\beta) & = 0 & \beta \in [-\tau_1, 0] \end{cases} \tag{3.16}$$

$$(\mathcal{K}_2) \begin{cases} \frac{\partial z_2}{\partial t}(x,t) & = \frac{\partial^2 z_2}{\partial x^2}(x,t) + k(x)w_2(t-\tau_2) &]0,a[\times]0,T[\\ z_2(0,t) & = z_2(a,t) = 0 &]0,T[\\ z_2(x,0) & = 0 &]0,a[\\ w_2(\beta) & = 0 & \beta \in [-\tau_2, 0] \end{cases} \tag{3.17}$$

admits a complete orthonormal system of eigenfunctions $(\eta_n)_{n\in\mathbb{N}^*}$ associated to the eigenvalues $\mu_n = -\frac{n^2\pi^2}{a^2}$, with $\eta_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x)$.

Each system $(\mathcal{K}_i)$ is augmented with the output equation corresponding to a sensor (Ω, e),

$$(\mathcal{E}_i)\; y_i(t) = \langle e, z_i(t)\rangle_{L^2(\Omega)}\,; 0 < t < T. \tag{3.18}$$

According to proposition(3.3), $(\mathcal{O}, l)$ dominates $(\mathcal{O}, h)$ with respect to the sensor (Ω, e), if and only if,

$$[\forall n \in \mathbb{N}^*, \langle l, \eta_n\rangle \langle e, \eta_n\rangle = 0] \Longrightarrow [\forall n \in \mathbb{N}^*, \langle k, \eta_n\rangle \langle e, \eta_n\rangle = 0]\,. \tag{3.19}$$

Let $m, n \in \mathbb{N}^*$ such that $m \neq n$. We suppose that $(\mathcal{K}_1)$ and $(\mathcal{K}_2)$ are respectively excited by the actuators $(\mathcal{O}, \eta_n)$ and $(\mathcal{O}, \eta_m)$. i.e $l = \eta_n$ and $k = \eta_m$.
Then

- $(\mathcal{O}, l)$ dominates $(\mathcal{O}, k)$ with respect to the sensor $(\mathcal{O}, \eta_n)$.
- $(\mathcal{O}, k)$ dominates $(\mathcal{O}, l)$ with respect to the sensor $(\mathcal{O}, \eta_m)$.

In the one-dimensional setting, any pair of operators D_1 and D_2 can be compared. However, this property does not generally hold in the two-dimensional case. Before presenting the numerical simulations, it is important to note that one can also discuss the domination property for a single system with two input operators. Such a system can be represented as follows:

$$(\mathcal{K}) \begin{cases} \dot{z}(t) = \mathcal{A}z(t) + D_1 w_{t,1} + D_2 w_{t,2}, \quad 0 < t < T, \\ z(0) = z_0 \in Z \end{cases} \tag{3.20}$$

and is augmented by the observation equation

$$y(t) = Qz(t). \tag{3.21}$$

and we have this remarks:

- Domination between input operators.
  In the case of a single system with two input operators, the analysis concerns the comparison between the control operators D_1 and D_2, rather than the comparison of two distinct systems $\mathcal{K}_1$ and $\mathcal{K}_2$.

- Definitions and properties preserved.
  All the definitions and structural properties related to domination continue to hold in this setting, with no modifications required.

- Domination as a consequence of remediability.
  For a single system, the notion of domination arises naturally as a direct consequence of the remediability concept (see [22]), which characterizes the ability to reconstruct the state through suitable control mechanisms.

- Minimal control via HUM.
  In this framework, the minimal control can be obtained by applying the Hilbert Uniqueness Method (HUM), as developed in reference [20]. This is precisely the approach that will be implemented in the numerical application presented in the following section.

3.3. Numerical Simulation

Now, we consider the following one-dimensional diffusion equation with delay acting only on the control input

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t) + l(x)u_1(t-\tau) + k(x)u_2(t-\tau), & (x,t) \in (0,1) \times (0,T), \\ z(0,t) = z(1,t) = 0, & 0 < t < T, \\ z(x,0) = z_0(x), & 0 < x < 1, \\ u_i(t) = 0, & t \in [-\tau_i, 0]. \end{cases}$$

The system is augmented by the output provided by a sensor (Ω, e), so that $Y = \mathbb{R}$, and

$$y(t) = Qz(t) = \langle e, z(t) \rangle_{L^2(\Omega)}, \quad 0 < t < T, \tag{2.29}$$

that is,

$$y(t) = Q\mathbb{S}(t)z_0 + \int_0^t Q\mathbb{S}(t-s)lu_1(s-\tau_1)\,ds + \int_0^t Q\mathbb{S}(t-s)ku_2(s-\tau_2)\,ds. \tag{2.30}$$

For the numerical simulations, we consider the case where the distributions of the actuator and the sensor are given by $l = e = k = \eta_1$. Thus $\Omega = \mathcal{O}$ and

$$\langle l, \eta_1 \rangle \langle e, \eta_1 \rangle = 1.$$

The corresponding (controlled) observation is given by

$$y_{u_1,u_2}(t) = \int_0^t e^{-\pi^2(t-s)} u_2(s-\tau_2) ds + \int_0^t e^{-\pi^2(t-s)} u_1(s-\tau_1) ds.$$

Consequently, the free observation ($u_1 = 0$) is expressed as

$$y_{0,u_2}(t) = \int_0^t e^{-\pi^2(t-s)} u_2(s) ds.$$

The optimal control ensuring the domination condition (see [22]) at the final time T is given by

$$u_1(s) = \langle l, \Psi_1^* \mathbb{S}^*(T-s) Q^* \xi_{u_2} \rangle_{L^2(\Omega)},$$

where

$$\xi_{u_2} = -\left(\Lambda_T\right)^{-1} QH_2u_2.$$

For $\xi \in \mathbb{R}$, the operator Λ_T is given by

$$\Lambda_T\xi = QH_1{H_1}^*Q^*\xi = \int_0^T Q\mathbb{S}(T-s)D_1\Psi_1\Psi_1^*D_1^*\mathbb{S}^*(T-s)Q^*\xi ds = \frac{1-e^{-2\pi^2T}}{2\pi^2}\xi.$$

The optimal control is then given by

$$u_1(s) = e^{-\pi^2(T-s-\tau_1)}\left(\frac{2\pi^2}{1-e^{-2\pi^2T}}\right)\int_0^T e^{-\pi^2(T-s)}u_2(s)ds.$$

For $T = 1$ and $\tau = 0.1$, numerical results are provided below for this choice of actuator: $u_2(t) = \psi(t)$ where $\psi(t) = t^2\exp(t)$. The objective is to confirm that the control $u_{\xi_{u_2}}$ compensate the effect of $u_2(t)$ (considered as perturbation) at the final time T.

After fixing $T = 1$ we get the following numerical simulation which perform the previous developments.

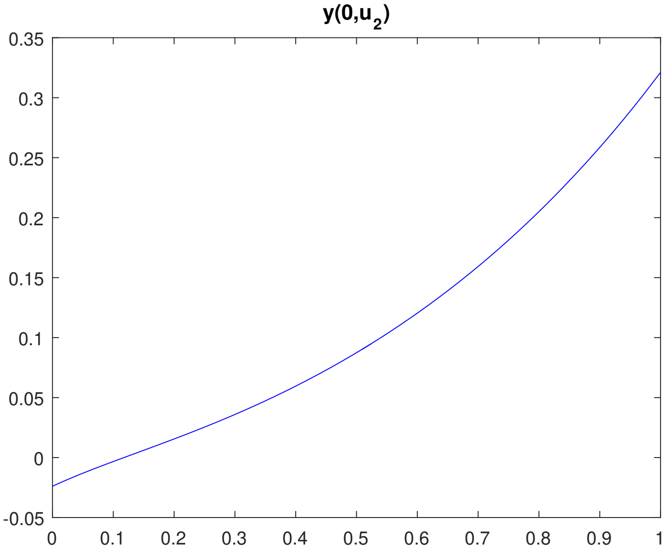

Figure 1: Representation of y_{0,u_2}

In the figures where no control is applied (or an "uncontrolled" trajectory is shown), we typically observe that the effect of the perturbation (modulated by delays) causes the system's output y_{0,u_2} to deviate significantly from its desired or normal state $y_{0,0}(t)$. For instance, if we see that the uncontrolled observation y_{0,u_2} diverges from the nominal response y_0, this divergence quantitatively illustrates that the disturbance operator (or its delayed version) is "injecting" energy (or error) into the state that is not being compensated.

When the optimized control u_1 (derived by considering the domination condition) is applied, our simulations illustrate that the resulting output y_{u_1,u_2} is brought close to or exactly to $y_{0,0}$, the nominal or undisturbed state, at the final time T. This convergence indicates that the control operator, even in the presence of delay, has sufficiently "covered" the directional influence of the disturbance operator.

4. Domination, reachability and cost reduction

In this section, the purpose is to give other applications and properties deriving from the domination concept. We respectively study with respect to the choice of the input operator B, problems of best

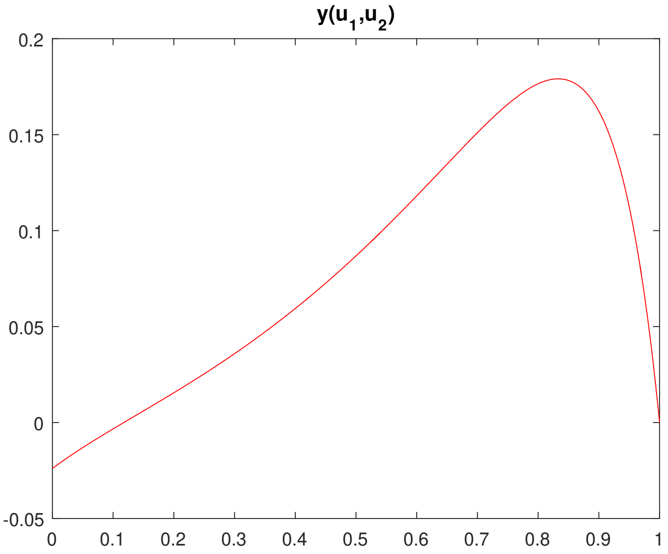

Figure 2: Representation of y_{u_1,u_2}

approach of a desired state and exact reachability with minimum energy, as well as the possibility of cost reduction in the case of actuators.
These illustrations concern some specific situations. However, the obtained results may be extended and adapted to other problems and more general cases.

We reconsider the system described by the following equation

$$(\mathcal{K})\begin{cases}\dot{x}(t)=\mathcal{A}x(t)+Dw(t-\tau); & 0<t<T\\ x(0)=x_0 & \\ w_1(\beta)=0 \quad \beta\in[-\tau,0]; & \tau\geq 0;\end{cases} \tag{4.1}$$

with the same hypothesis and notations. The solution of the delayed state equations $\mathcal{K}$ is given by

$$x(t)=\mathbb{S}(t)x_0+\int_0^t \mathbb{S}(t-s)D\Psi(s)wds.$$

We consider the operators

$$\begin{array}{rl} H(B):L^2(0,T;U)\rightarrow & X\\ w\rightarrow & \int_0^T \mathbb{S}(T-s)D\Psi(s)wds.\end{array}$$

Consequently, the corresponding output at the final time T, is given by

$$x(T)=\mathbb{S}(T)x_0+\int_0^T \mathbb{S}(T-s)D\Psi(s)wds. \tag{4.2}$$

In what follows, we assume without loss of generality, that the initial state $x_0=0$ and that the output operator Q = I. Hence, we have

$$y_i(T)=x_i(T)=H(D)w=\int_0^T \mathbb{S}(T-s)D\Psi(s)wds.$$

4.1. Domination and best approach of a desired state

In this part, we show that the concept of domination makes it possible to improve the approach of a desired state. More precisely, we have the following property

Proposition 4.1 *A dominating control operator ensures a better approach of any desired state $x_d \in X$.*

Proof: Indeed, let D_1 and D_2 be two input operators such that $D_i \in \mathcal{L}(W_i, X)$ for $i = \{1, 2\}$, where W_1, W_2, are two Hilbert spaces.

For a desired state $x_d \in X$, x_d not necessarily reachable exactly, we consider the functions J_1 and J_2 respectively defined by:

$$J_1(w_1) = \|H(D_1)w_1 - x_d\|_X^2 \, ; \; w_1 \in L^2(0, T; W_1) \tag{4.3}$$

and

$$J_2(w_2) = \|H(D_2)w_2 - x_d\|_X^2 \, ; \; w_2 \in L^2(0, T; W_2) \,. \tag{4.4}$$

- We assume that the operator D_2 dominates D_1, exactly, i.e.

$$Im(H(D_1)) \subset Im(H(D_2))$$

we have

$$\begin{aligned} \inf_{w_2 \in L^2(0,T;W_2)} J_2(w_2) &= \inf_{z_2 \in \mathrm{Im}(H(D_2))} \|z_2 - z_d\|_Z^2 \\ &\leq \inf_{z_1 \in \mathrm{Im}(H(D_1))} \|z_1 - z_d\|_Z^2 = \inf_{w_1 \in L^2(0,T;W_1)} J_1(w_1) \,. \end{aligned}$$

Hence,

$$\inf_{w_2 \in L^2(0,T;W_2)} J_2(w_2) \leq \inf_{w_1 \in L^2(0,T;W_1)} J_1(w_1) \,.$$

Consequently, D_2 ensures a better approach of the desired state x_d which is not necessarily reachable exactly, i.e. even if the corresponding system is not exactly controllable.

□

- Let us note that this result remain practically the same in the case where D_2 dominates D_1 weakly.

In the particular case where $Im(H(D_2)) = X$ (respectively $\overline{Im(H(D_2))} = X$), or equivalently, the corresponding system is exactly (respectively weakly) controllable, the result ramain true for every control space W_1 and any input operator $D_1 \in \mathcal{L}(W_1, X)$. In such a situation, we have

$$\inf_{w_2 \in L^2(0,T;W_2)} J_2(w_2) = 0.$$

4.2. Domination and exact reachability with minimum energy

In this section, we examine the relationship between the concept of domination and the problem of exact reachability with minimum energy.
We assume without loss of generality, that the input operator $D \in \mathcal{L}(W, X)$ is of the form

$$Dw = D_1 w_1 + D_2 w_2$$

where $D_1 \in \mathcal{L}(W_1, X)$, $D_2 \in \mathcal{L}(W_2, X)$, $w \in L^2(0,T;U)$, $w_1 \in L^2(0,T;W_1)$, $w_2 \in L^2(0,T;W_2)$; W_1 and W_2 are subspaces of the control space W such that:

$$W = W_1 \oplus W_2.$$

Thus, for $w_1 + w_2 \in L^2(0,T;W)$ with $w_i \in L^2(0,T;W_i)$; $i \in \{1,2\}$, we have

$$H(D)u = H(D_1)\, w_1 + H(D_2)\, w_2 \tag{4.5}$$

which may be written

$$H(D)w = (H(D_1)\, H(D_2)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Obviously,

$$\mathrm{Im}(H(D)) = \mathrm{Im}(H(D_1)) + \mathrm{Im}(H(D_2)) \tag{4.6}$$

then B dominates exactly D_1, as well as D_2.

Let x_d be a desired state such that $x_d \in \mathrm{Im}(H(D_1))$, consequently $x_d \in \mathrm{Im}(H(B))$.
Hence, both operators allow to reach exactly the state x_d.

We have the following result concerning the problem of exact reachability of the desired state x_d, with minimum energy.

Proposition 4.2 *The dominating control operator D, ensures a reduced cost with respect to D_1.*

Proof: In order to establish this minimum energy result, we consider

- the corresponding sets of admissible controls defined as follows

$$\mathcal{W}_{ad} = \left\{ w \in L^2(0,T;W); H(D)w = x_d \right\}$$

and

$$\mathcal{W}^1_{ad} = \left\{ w_1 \in L^2(0,T;W_1); H(D_1)\, w_1 = x_d \right\}$$

$\mathcal{W}_{ad}$ and $\mathcal{W}^1_{ad}$ are non empty, convex and closed subsets of $L^2(0,T;W)$ and $L^2(0,T;W_1)$ respectivly.

- the cost functions J and J_1 defined by:

$$J(w) = \|w\|^2_{L^2(0,T;W)};\ w \in L^2(0,T;W) \tag{4.7}$$

and

$$J_1(w_1) = \|w_1\|^2_{L^2(0,T;W_1)};\ w_1 \in L^2(0,T;W_1). \tag{4.8}$$

Hence, for $w_1 \in \mathcal{W}^1_{ad}$, the control $w \equiv \left(\begin{matrix} w_1 \\ w_2 \end{matrix} \right) \in \mathcal{W}_{ad}$, then $\mathcal{W}^1_{ad}$ is a subset of $\mathcal{W}_{ad}$.
Moreover, we have $\|w_1\|^2_{L^2(0,T;W_1)} = \|w\|^2_{L^2(0,T;W)}$, therefore

$$J_1(w_1) = J(w) \tag{4.9}$$

and

$$\inf_{w \in \mathcal{W}_{ad}} J(w) \leq \inf_{w_1 \in \mathcal{W}_{ad}^1} J_1\left(w_1\right). \tag{4.10}$$

Consequently, the minimal cost associated to the operator D, is reduced with respect to that associated to D_1. $\square$

5. Conclusion

This paper investigated the domination problem for a class of linear distributed systems with delays. Within the framework of strongly continuous semigroups, we established conditions under which the effect of delayed states can be dominated by suitable control operators. A classification of these admissible control operators has been proposed and analyzed in detail. As an illustration, we applied our results to a diffusion-type system with delayed feedback, showing how the theoretical framework can be used in a concrete setting. The approach provides a structured method to study the influence of delays and the role of control operators in shaping system behavior. The considered approach open several perspectives for future research, including extensions to nonlinear or stochastic systems and the consideration of other classes of delays.

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