Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 18** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.82729

The Nehari Manifold for a Fractional $(p(x,.),q(x,.))-$Laplacian Elliptic System

Athmane Boumazourh, Houria El-Yahyaoui and Mohammed Shimi

ABSTRACT: In this paper, we study the existence and multiplicity of weak solutions for a class of nonlocal elliptic systems involving fractional $(p(x,.),q(x,.))$-Laplacian operators under Neumann boundary conditions. The problem is formulated in fractional Sobolev spaces with variable exponents, where the interaction between nonlocality and space-dependent growth leads to an energy functional that is not lower bounded on the associated functional space. To overcome this difficulty, we use the Nehari manifold approach, which allows us to recover a natural constraint on which the functional becomes coercive and bounded from below. Our results extend recent advances in the field of fractional elliptic problems with variable exponents and provide new insights into nonlocal systems subject to Neumann boundary conditions.

Keywords: Fractional Sobolev spaces, Fractional $p(x,.)$-Laplacian operator, Nehari manifold, Neumann boundary conditions.

Contents

1. Introduction

In this paper, we study the existence of positive solutions for the following fractional elliptic system

$$\left\{\begin{array}{lll} (-\Delta)^s_{p(x,.)}(u)+|u|^{\bar{p}(x)-2}u=\lambda\,|u|^{r(x)-2}u+\frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & in & \Omega\\ (-\Delta)^s_{q(x,.)}(v)+|v|^{\bar{q}(x)-2}v=\mu\,|v|^{r(x)-2}v+\frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & in & \Omega\\ \mathcal{N}^s_{p(x,.)}u=\mathcal{N}^s_{q(x,.)}v=0, & in & \mathbb{R}^N\backslash\Omega \end{array}\right. \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $(N>2)$, λ, $\mu>0$ are two parameters. Let $p,q:\overline{Q}\longrightarrow]1,+\infty[$ be continuous bounded functions, $r\in C(\overline{\Omega})$ and $\alpha>1$, $\beta>1$ such that

$$\begin{aligned} 1<r^-\leq r(x)\leq r^+<p^-\leq p(x,y)\leq p^+<\alpha+\beta<p^*_s(x)\\ 1<r^-\leq r(x)\leq r^+<q^-\leq q(x,y)\leq q^+<\alpha+\beta<q^*_s(x) \end{aligned} \tag{1.2}$$

and

$$(\alpha+\beta)\left(\min\{p^-,q^-\}-r^+\right)>\min\{p^-,q^-\}\left(\max\{p^+,q^+\}-r^-\right) \tag{1.3}$$

where

$$p^-=\inf_{x,y\in Q}p(x,y),\qquad p^+=\sup_{x,y\in Q}p(x,y) \tag{1.4}$$

$$q^-=\inf_{x,y\in Q}q(x,y),\qquad q^+=\sup_{x,y\in Q}q(x,y) \tag{1.5}$$

$$r^-=\inf_{x\in\Omega}r(x),\qquad r^+=\sup_{x\in\Omega}r(x). \tag{1.6}$$

and $Q=(\mathbb{R}^{\mathbb{N}}\times\Omega)\cup(\Omega\times\mathbb{R}^{\mathbb{N}})$. The critical fractional Sobolev exponent is given by

$$m^*_s(x)=\left\{\begin{array}{ll} \frac{Nm(x,x)}{N-sm(x,x)} & if\quad N>sm(x,x)\\ +\infty & if\quad N\leqslant sm(x,x) \end{array}\right.$$

2020 *Mathematics Subject Classification:* 35B09, 35J50, 35R11.
Submitted March 16, 2026. Published June 19, 2026.

For $s \in (0,1)$, $(-\Delta)^s_{p(x,.)}$ is the fractional $p(x,.)-$Laplacian operator which is defined as

$$(-\Delta)^s_{p(x,.)}u(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy, \quad \forall x\in\mathbb{R}^N \tag{1.7}$$

and the operator $\mathcal{N}^s_{p(x,.)}$ represents the nonlocal normal $p(x,.)-$ derivative introduced in [5] and defined by

$$\mathcal{N}^s_{p(x,.)}u = \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy, \quad \forall x\in\mathbb{R}^n\setminus\Omega, \tag{1.8}$$

Recently, the theory of fractional Lebesgue and Sobolev spaces and their extensions to the variable exponent setting has witnessed remarkable progress, motivated both by intrinsic mathematical interest and by numerous applications in science and engineering (see [6,14,17]). These spaces generalize the classical fractional framework by allowing the integrability and differentiability exponents to vary with respect to the spatial variable, thus providing a flexible and powerful tool for modeling heterogeneous media and nonstandard growth phenomena. From an analytical point of view, the lack of homogeneity and translation invariance in the variable exponent context leads to substantial difficulties, particularly concerning compactness properties, embedding theorems, and variational methods.

In parallel, nonlocal operators such as the fractional p(x,.)-Laplacian have become central objects of investigation in nonlinear analysis. Problems involving this operator and the corresponding nonlocal elliptic equations represent a relatively new and rapidly evolving research direction, combining the challenges of nonlocality with those arising from variable exponent nonlinearities. As a result, many classical tools must be carefully adapted or extended, which has stimulated a growing body of literature devoted to qualitative properties of solutions, including existence, multiplicity, regularity, and asymptotic behavior (see, for example, [1,2,4,5,9,14]).

Fractional p(x,.)-Laplacian type problems naturally arise in a wide range of physical and applied contexts where long-range interactions, memory effects, or spatial heterogeneities cannot be neglected. Typical applications include models from conservation laws, ultra-materials, and water wave propagation, as well as problems in population dynamics, optimization, and mathematical finance. Further relevant applications appear in the study of soft thin films, stratified and composite materials, phase transition phenomena, anomalous diffusion processes, semipermeable membranes, crystal dislocation theory, flame propagation, and ultra-relativistic limits of quantum mechanics. For further background and motivation on these applications, we refer the reader to [7,17].

In the constant case and under Dirichlet boundary conditions as will as if $p=q$, the authors have studied in [8] the following problem

$$\begin{cases} (-\Delta)^s_p(u) = \lambda\,|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & in \quad \Omega \\ (-\Delta)^s_p(u) = \mu\,|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & in \quad \Omega \\ u=v=0, & in \quad \mathbb{R}^N\setminus\Omega. \end{cases} \tag{1.9}$$

where Ω is a smooth bounded set in $\mathbb{R}^N$, $N>ps$ with $s\in(0,1)$ fixed, $\lambda,\mu>0$ are two parameters, $1<r<p$ and $\alpha,\ \beta>1$ satisfy $p<\alpha+\beta<p^*_s$. $p^*_s=\frac{Np}{N-sp}$ is the critical fractional Sobolev exponent.
In the local case s=1 Hsu [13] considered the following p-Laplacian system

$$\begin{cases} (-\Delta)_p(u) = \lambda\,|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & in \quad \Omega \\ (-\Delta)_p(u) = \mu\,|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & in \quad \Omega \\ u=v=0, & on \quad \partial\Omega. \end{cases} \tag{1.10}$$

where $1<r<p$, $\alpha,\ \beta>1$ and $\alpha+\beta=p^*$.
Another work is concerned on the study of the following elliptic system involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s(u) = \lambda\,|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & in \quad \Omega \\ (-\Delta)^s(u) = \mu\,|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & in \quad \Omega \\ u=v=0, & in \quad \mathbb{R}^N\setminus\Omega. \end{cases} \tag{1.11}$$

Using variational methods and a Nehari manifold decomposition X. He, M. Squassina, and W. Zou [12] have proved that (1.11) admits at least two positive solutions.
On the other hand, recent years have witnessed growing interest in fractional nonlocal problems under Neumann boundary conditions, investigated by means of various methods (see [19]). In this framework, Azroul et al in [3] have guaranteed the existence of weak solutions for the following fractional elliptic system using the Mountain Pass Theorem

$$(\mathcal{P})\begin{cases} (-\Delta)^s_{p(x,.)}(u) + |u|^{\bar{p}(x)-2}u = F_u(x,u,v) & \text{in } \Omega, \\ (-\Delta)^s_{q(x,.)}(v) + |v|^{\bar{q}(x)-2}v = F_v(x,u,v) & \text{in } \Omega, \\ \mathcal{N}^s_{p(x,.)}u = G_u(x,u,v) & \text{in } \mathbb{R}^N \setminus \Omega, \\ \mathcal{N}^s_{q(x,.)}v = G_v(x,u,v), & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is an open bounded subset in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary $\partial\Omega$, $s \in (0,1)$, $p,q : \overline{Q} \to (1,+\infty)$ are symmetric, continuous and bounded functions.
The main objective of this work is to establish the existence of positive solutions for the considered fractional system under Neumann boundary conditions by means of the Nehari manifold technique. The result is formulated as follows.

Theorem 1.1 *Let $s \in (0,1)$, $N > 2$ and let Ω be an open bounded subset of $\mathbb{R}^N$, then there exists a constant $C_{\lambda,\mu} > 0$ such that for $0 < \max\{\lambda,\mu\} < C_{\lambda,\mu}$, problem (1.1) has at least two nontrivial positive solutions.*

This paper is organized as follows. In section 2, we recall some notations and properties of fractional Lebesgue and functional setting. In order to prove our main Theorem in section 3 some usable Lemmas are given.

2. Preliminaries and functional setting

In this section, we present the fractional setting and recall several essential definitions and proprties of generalized Lebesgue spaces. For more details we refer to [11,15,18], and the references therein.
Consider the set

$$C_+(\overline{\Omega}) = \{m \in C(\overline{\Omega}) : m(x) > 1, \forall x \in \overline{\Omega}\}.$$

For any $m \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{m(x)}(\Omega)$ as

$$L^{m(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{m(x)}dx < +\infty\right\}.$$

This space equipped with the *Luxemburg* norm

$$\|u\|_{L^{m(x)}(\Omega)} = inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{m(x)} dx \leqslant 1\right\}$$

is a separable reflexive Banach space.
Let $m' \in C_+(\overline{\Omega})$ be the conjugate exponent of m, i.e., $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1 *(Hölder inequality). If $u \in L^{m(x)}(\Omega)$ and $v \in L^{m'(x)}(\Omega)$, so*

$$\left|\int_\Omega uvdx\right| \leqslant \left(\frac{1}{m^-} + \frac{1}{m'^-}\right)\|u\|_{L^{m(x)}(\Omega)}\|v\|_{L^{m'(x)}(\Omega)} \leqslant 2\|u\|_{L^{m(x)}(\Omega)}\|v\|_{L^{m'(x)}(\Omega)}$$

The modular of $L^{m(x)}(\Omega)$ is defined by

$$\begin{aligned} \rho_{m(.)} : L^{m(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow \rho_{m(.)}(u) = \textstyle\int_\Omega |u(x)|^{m(x)}dx \end{aligned}$$

Proposition 2.1 *[10,15] Let $u \in L^{m(x)}(\Omega)$, then we have,*

1. $\|u\|_{L^{(x)}(\Omega)} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho_{m(.)}(u) < 1$ *(resp* $= 1, > 1$*),*
2. $\|u\|_{L^{m(x)}(\Omega)} < 1 \Rightarrow \|u\|^{m+}_{L^{m(x)}(\Omega)} \leqslant \rho_{m(.)}(u) \leqslant \|u\|^{m-}_{L^{m(x)}(\Omega)}$,
3. $\|u\|_{L^{m(x)}(\Omega)} > 1 \Rightarrow \|u\|^{m-}_{L^{m(x)}(\Omega)} \leqslant \rho_{m(.)}(u) \leqslant \|u\|^{m+}_{L^{m(x)}(\Omega)}$.

Proposition 2.2 *If $u, u_k \in L^{m(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

1. $\lim\limits_{k\to+\infty} \|u_k - u\|_{L^{m(x)}(\Omega)} = 0$,
2. $\lim\limits_{k\to+\infty} \rho_{m(.)}(u_k - u) = 0$,
3. $u_k \longrightarrow u$ *in measure in* Ω *and* $\lim\limits_{k\to+\infty} \rho_{m(.)}(u_k) = \rho_{m(.)}(u)$.

Proposition 2.3 *[10] Let Ω be an open bounded subset of $\mathbb{R}^N$, $m \in C(\bar{\Omega})$, then $(L^{m(x)}(\Omega), ||u||_{L^{(x)}(\Omega)})$ is a reflexive uniformly convex and separable Banach space.*

Now, let introduce our fundamental space. Let $m : \overline{Q} \to (1, +\infty)$ is a continuous, symmetric and bounded function and let $sm^+ < N$;
We set

$$X^{m(x,y)} := \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \|u\|_{X^{m(x,y)}} < 1\right\}$$

where:

$$\|u\|_{X^{m(x,y)}} = [u]_{s,\bar{m}(.),Q} + \|u\|_{\bar{m}(.)}$$

and

$$[u]_{s,\bar{m}(.),Q} = \inf\left\{\lambda \geq 0 : \int_Q \frac{|u(x)-u(y)|^{m(x,y)}}{2\lambda^{m(x,y)}|x-y|^{N+sm(x,y)}}dxdy \leq 1\right\}$$

Proposition 2.4 *[3] Let $m : \overline{Q} \to (1, +\infty)$ is a continuous, symmetric and bounded function satisfying (1.2), $s \in (0, 1)$ and $sm^+ < N$, then $(X^{m(x,y)}, \|u\|_{X^{m(x,y)}})$ is a reflexive Banach space.*

For any $u \in X^{m(x,y)}$, we define the functional

$$\rho_{s,m(.,.),Q}(u) = \int_Q \frac{|u(x)-u(y)|^{m(x,y)}}{2|x-y|^{N+sm(x,y)}}dxdy + \int_\Omega |u|^{\bar{m}(x)}dx,$$

it is easy to see that $\rho_{s,m(.,.),Q}$ is a convex modular on $X^{m(x,y)}$. The norm associated with $\rho_{s,m(.,.),Q}$ is given by

$$\|u\| = \inf\left\{\lambda > 0 \ : \rho_{s,m(.,.),Q}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Note that the norm $\|.\|$ is equivalent on $X^{m(x,y)}$ to the norm $\|.\|_{X^{m(x,y)}}$.
We could also get the following properties:

Lemma 2.2 *[3] Let $m : \overline{Q} \to (1, +\infty)$ is a continuous, symmetric and bounded function satisfies (1.2), $s \in (0, 1)$ and $sm^+ < N$, and let $u \in X^{m(x,y)}$. Then, the following assertions hold:*

1. $\|u\|_{X^{m(x,y)}} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho_{s,m(.,.),Q}(u) < 1$ *(resp* $= 1, > 1$*).*
2. $\|u\|_{X^{m(x,y)}} > 1 \Rightarrow \|u\|^{m^-}_{X^{m(x,y)}} \leq \rho_{s,m(.,.),Q}(u) \leq \|u\|^{m^+}_{X^{m(x,y)}}$,
3. $\|u\|_{X^{m(x,y)}} < 1 \Rightarrow \|u\|^{m^+}_{X^{m(x,y)}} \leq \rho_{s,m(.,.),Q}(u) \leq \|u\|^{m^-}_{X^{m(x,y)}}$.

4. $\rho_{s,m(.,.),Q}(u_k - u) \longrightarrow_{k\to+\infty} 0 \Leftrightarrow \|u_k - u\|_{X^{m(x,y)}} \longrightarrow_{k\to+\infty} 0$

Theorem 2.1 *[3] Let Ω be a Lipschitz bounded domain in $\mathbb{R}^n$. Let $s \in (0,1)$, $m : \overline{Q} \to (1,+\infty)$ is a continuous, symmetric and bounded function satisfies (1.2) and $sm^+ < N$. If $r : \Omega \to (1,+\infty)$ be a continuous variable exponent such that*

$$1 < r^- \leq r(x) < m_s^*(x) \text{ for all } x \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N,s,m,r,\Omega) > 0$ such that for any $u \in X^{m(x,y)}$,

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{X^{m(x,y)}}.$$

That is, the space $X^{m(x,y)}$ is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

Lemma 2.3 *Let*

$$H_{s,p(.,.),Q}(u) = \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^{m(x,y)}}{m(x,y)|x-y|^{N+sm(x,y)}}dxdy + \int_\Omega \frac{|u(x)|^{\bar{m}(x)}}{\bar{m}(x)}dx,$$

then, we have the following properties:

1. *The function $H_{s,m(.,.),Q}$ is of class $C^1(X^{m(x,y)},\mathbb{R})$.*

2. *The function $H'_{s,m(.,.),Q}$ is coercive.*

3. *The function $H'_{s,m(.,.),Q}$ is strictly monotone operator.*

4. *$H'_{s,m(.,.),\mathbb{R}^{2N}}$ is a mapping of type (S^+), that is, if $u_n \rightharpoonup u$ in $X^{m(x,y)}$ and $\limsup_{n\to\infty}\left\langle H'_{s,m(.,.),\mathbb{R}^{2N}}(u_n), u_n - u\right\rangle \leq 0$, then $u_n \to u$ strongly in $X^{m(x,y)}$.*

We next present the analogue of the divergence theorem and the integration by parts formula in the nonlocal setting.

Proposition 2.5 *[3] Let $s \in (0,1)$ and $m : \overline{Q} \to (1,+\infty)$ is a continuous, symmetric and bounded function satisfies (1.2) with $sm^+ < N$ and let u be any bounded C^2 function in $\mathbb{R}^N$. Then,*

$$\int_\Omega (-\Delta)^s_{m(.,.)}u(x)dx = -\int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_{s,m(.,.)}u(x)dx.$$

Proposition 2.6 *[3] Let $s \in (0,1)$ and $m : \overline{Q} \to (1,+\infty)$ is a symmetric, continuous and bounded function satisfies (1.2) with $sm^+ < N$ and let u et φ be any bounded C^2 function in $\mathbb{R}^N$. Then,*

$$\begin{aligned}\tfrac{1}{2}\textstyle\int_Q &\frac{|u(x)-u(y)|^{m(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sm(x,y)}}dxdy \\ &= \textstyle\int_\Omega \varphi(-\Delta)^s_{m(.,.)}u(x)dx + \int_{\mathbb{R}^N\setminus\Omega}\varphi\mathcal{N}_{s,m(.,.)}u(x)dx.\end{aligned}$$

Let $E = X^{p(x,y)} \times X^{q(x,y)}$ be the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\|(u,v)\| = \|u\|_{X^{p(x,y)}} + \|v\|_{X^{q(x,y)}}. \tag{2.1}$$

3. Main results

Definition 3.1 *We say that* $(u,v)\in E$ *is a weak solution of* (1.1) *if*

$$\frac{1}{2}\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega |u|^{\bar{p}(x)-2}u\varphi dx$$
$$+\frac{1}{2}\int_Q \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega |v|^{\bar{q}(x)-2}v\psi dx$$

$$=\int_\Omega \big(\lambda|u|^{r(x)-2}u\varphi+\mu|v|^{r(x)-2}v\psi\big)dx+\frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^{\beta}\varphi dx+\frac{2\beta}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta-2}v\psi dx.$$

for all $(\varphi,\psi)\in E$.

The fact that $(u,v)\in E$ is a weak solution of (1.1) is equivalent to being a critical point of the following functional

$$J_{\lambda,\mu}(u,v)=\frac{1}{2}\int_Q \frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega \frac{|u|^{\bar{p}(x)}}{\bar{p}(x)}dx$$
$$+\frac{1}{2}\int_Q \frac{1}{q(x,y)}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega \frac{|v|^{\bar{q}(x)}}{\bar{q}(x)}dx$$

$$-\int_\Omega \frac{1}{r(x)}\big(\lambda|u|^{r(x)}+\mu|v|^{r(x)}\big)dx-\frac{2}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta}dx.$$

We have that $J_{\lambda,\mu}\in C^1(E,\mathbb{R})$ and

$$< J'_{\lambda,\mu}(u,v),(\varphi,\psi) > =\frac{1}{2}\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega |u|^{\bar{p}(x)-2}u(x)\varphi(x)dx$$
$$+\frac{1}{2}\int_Q \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega |v|^{\bar{q}(x)-2}v(x)\psi(x)dx$$
$$-\int_\Omega \big(\lambda|u|^{r(x)-2}u\varphi+\mu|v|^{r(x)-2}v\psi\big)dx-\frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^{\beta}\varphi dx$$
$$-\frac{2\beta}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta-2}v\psi dx.$$

for any $(\varphi,\psi)\in E$.
According to (1.2), it follows that the functional $J_{\lambda,\mu}$ is unbounded from below on E, but it becomes bounded from below when restricted to suitable subset of E, namely the Nehari manifold associated with $J_{\lambda,\mu}$, defined as

$$\mathcal{N}_{\lambda,\mu}=\{(u,v)\in E\backslash\{(0,0)\} :< J'(u,v),(u,v) >= 0\}.$$

Let

$$I_{\lambda,\mu}(u,v) =< J'_{\lambda,\mu}(u,v),(u,v) >$$

$$=\frac{1}{2}\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega |u(x)|^{\bar{p}(x)}dx$$
$$+\frac{1}{2}\int_Q \frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega |v(x)|^{\bar{q}(x)}dx$$

$$-\int_\Omega \left(\lambda|u|^{r(x)}+\mu|v|^{r(x)}\right)dx-2\int_\Omega |u|^{\alpha}|v|^{\beta}dx \tag{3.1}$$

Then for $(u,v)\in\mathcal{N}_{\lambda,\mu}$ we have

$$\begin{aligned}
< I'_{\lambda,\mu}(u,v),(u,v)> &= \frac{1}{2}\int_Q p(x,y)\tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega \bar{p}(x)|u(x)|^{\bar{p}(x)}dx \\
&\quad +\frac{1}{2}\int_Q q(x,y)\tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega \bar{q}(x)|v(x)|^{\bar{q}(x)}dx \qquad (3.2)\\
&\quad -\int_\Omega r(x)\Big(\lambda|u|^{r(x)}+\mu|v|^{r(x)}\Big)dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx \\
&\leqslant \max\{p^+,q^+\}\Big[\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx + 2\int_\Omega |u|^\alpha|v|^\beta dx\Big] \\
&\quad -r^-\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx \\
&\leqslant \left(\max\{p^+,q^+\}-r^-\right)\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx \\
&\quad +(\max\{p^+,q^+\}-2(\alpha+\beta))\int_\Omega |u|^\alpha|v|^\beta dx.
\end{aligned}$$

Now, we split $\mathcal{N}_{\lambda,\mu}$ into three parts

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} \ :\ < I'_{\lambda,\mu}(u,v),(u,v)>> 0\}\\
\mathcal{N}^-_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} \ :\ < I'_{\lambda,\mu}(u,v),(u,v)>< 0\}\\
\mathcal{N}^0_{\lambda,\mu} &= \{(u,v)\in\mathcal{N}_{\lambda,\mu} \ :\ < I'_{\lambda,\mu}(u,v),(u,v)>= 0\}
\end{aligned}$$

Lemma 3.1 *The energy functional $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof: Suppose that $\|u\|_{X^{p(x,y)}}>1$ and $\|v\|_{X^{q(x,y)}}>1$. From the definition of $J_{\lambda,\mu}$ we have

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &\geqslant \tfrac{1}{p^+}\left(\frac{1}{2}\int_Q \tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx\right)\\
&\quad +\tfrac{1}{q^+}\left(\frac{1}{2}\int_Q \tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v(x)|^{\bar{q}(x)}dx\right)\\
&\quad -\tfrac{1}{r^-}\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx - \tfrac{1}{\alpha+\beta}\Bigg(\frac{1}{2}\int_Q \tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx\\
&\quad +\frac{1}{2}\int_Q \tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v(x)|^{\bar{q}(x)}dx - \int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx\Bigg)\\
&\geqslant \left(\min\{\tfrac{1}{p^+},\tfrac{1}{q^+}\}-\tfrac{1}{\alpha+\beta}\right)\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)) + (\tfrac{1}{\alpha+\beta}-\tfrac{1}{r^-})\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx\\
&\geqslant \left(\min\{\tfrac{1}{p^+},\tfrac{1}{q^+}\}-\tfrac{1}{\alpha+\beta}\right)\left(\|u\|^{p^-}_{X^{p(x,y)}}+\|v\|^{q^-}_{X^{q(x,y)}}\right) + (\tfrac{1}{\alpha+\beta}-\tfrac{1}{r^-})\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx
\end{aligned}$$

Since $r(x)<p^*(x)$ and $r(x)<q^*(x)$, we have the continuous embedding of $X^{p(x,y)}$ in $L^{r(x)}$ and of $X^{q(x,y)}$ in $L^{r(x)}$. Then we get

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &\geqslant \left[\min\{\tfrac{1}{p^+},\tfrac{1}{q^+}\}-\tfrac{1}{\alpha+\beta}\right]||(u,v)||_E^{\min\{p^-,q^-\}}\\
&\quad +c(\tfrac{1}{\alpha+\beta}-\tfrac{1}{r^-})\max\{\lambda,\mu\}||(u,v)||_E^{r^+}
\end{aligned}$$

Since $\min\{p^-,q^-\}>r^+$, we have $J_{\lambda,\mu}(u,v)\longrightarrow\infty$ as $||(u,v)||_E\longrightarrow\infty$. This implies that $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$. □

Lemma 3.2 *There exists a constant $C_{\lambda,\mu} > 0$ such that for $0 < \max\{\lambda,\mu\} < C_{\lambda,\mu}$ we have $\mathcal{N}_{\lambda,\mu} = \emptyset$.*

Proof: Suppose that $\mathcal{N}^0_{\lambda,\mu}(\Omega) \neq \emptyset$ for all $\lambda,\mu \in \mathbb{R}\backslash\{0\}$.
Let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}(\Omega)$ such that $\|u\|_{X^{p(x,y)}} > 1$ and $\|v\|_{X^{q(x,y)}} > 1$. We have

$$\begin{aligned}
0 &=< I'_{\lambda,\mu}(u,v),(u,v) > \\
&= \frac{1}{2}\int_Q p(x,y)\tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega \bar{p}(x)|u(x)|^{\bar{p}(x)}dx \\
&\quad +\frac{1}{2}\int_Q q(x,y)\tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega \bar{q}(x)|v(x)|^{\bar{q}(x)}dx \\
&\quad -\int_\Omega r(x)(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx \\
&\geqslant \min\{p^-,q^-\}\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v)\right) - r^+\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v) - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx\right) \\
&\geqslant (\min\{p^-,q^-\} - r^+)\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v)\right) - 2(\alpha+\beta-r^+)\int_\Omega |u|^\alpha|v|^\beta dx,
\end{aligned}$$

by the Young inequality we get

$$\begin{aligned}
&0 =< I'_{\lambda,\mu}(u,v),(u,v) > \\
&\geqslant (\min\{p^-,q^-\} - r^+)\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v)\right) - \tfrac{2\alpha(\alpha+\beta-r^+)}{\alpha+\beta}\int_\Omega |u|^{\alpha+\beta}dx \\
&\qquad -\tfrac{2\beta(\alpha+\beta-r^+)}{\alpha+\beta}\int_\Omega |v|^{\alpha+\beta}dx.
\end{aligned}$$

Using the compact embedding of $X^{p(x,y)}$ and $X^{q(x,y)}$ in $L^{\alpha+\beta}(\Omega)$, there exist c_p and c_q such that

$$\begin{aligned}
0 = \quad & < I'_{\lambda,\mu}(u,v),(u,v) > \\
& \geqslant (\min\{p^-,q^-\} - r^+)\left(\|u\|^{p^-}_{X^{p(x,y)}} + \|v\|^{q^-}_{X^{q(x,y)}}\right) - \tfrac{2(\alpha+\beta-r^+)}{\alpha+\beta}\min\{c_p\alpha, c_q\beta\}(\|u\|^{\alpha+\beta}_{X^{p(x,y)}} \\
&\qquad +\|v\|^{\alpha+\beta}_{X^{q(x,y)}})dx,
\end{aligned}$$

thus we get

$$\|(u,v)\|_E \geqslant \left(\frac{(\alpha+\beta)(\min\{p^- + q^-\} - r^+)}{2[(\alpha+\beta) - r^+]\min\{c_p\alpha, c_q\beta\}}\right)^{\frac{1}{\alpha+\beta-\min\{p^-,q^-\}}} \tag{3.3}$$

On the other hand we have

$$
\begin{aligned}
0 & =< I'_{\lambda,\mu}(u,v),(u,v) > \\
& = \frac{1}{2}\int_Q p(x,y)\tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega \bar{p}(x)|u|^{\bar{p}(x)}dx \\
& \quad + \frac{1}{2}\int_Q q(x,y)\tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega \bar{q}(x)|v|^{\bar{q}(x)}dx \\
& \quad - \int_\Omega r(x)(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx - 2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx \\
& \leqslant \max\{p^+,q^+\}\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v)\right) - r^-\int_\Omega(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx \\
& \qquad -(\alpha+\beta)\left[\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v) - \int_\Omega(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx\right] \\
& \leqslant (\max\{p^+,q^+\} - (\alpha+\beta))\left(\rho_{s,p(x,y),Q}(u) + \rho_{s,q(x,y),Q}(v)\right) \\
& \qquad +(\alpha+\beta-r^-)\int_\Omega(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx \\
& \leqslant (\max\{p^+,q^+\} - (\alpha+\beta))\left(\|u\|^{p^+}_{X^{p(x,y)}} + \|v\|^{q^+}_{X^{q(x,y)}}\right) \\
& \qquad +(\alpha+\beta-r^-)(\lambda c_p\|u\|^{r^+}_{X^{p(x,y)}} + \mu c_q\|v\|^{r^+}_{X^{p(x,y)}}) \\
& \leqslant (\max\{p^+,q^+\} - (\alpha+\beta))\|(u,v)\|_E^{\max\{p^+,q^+\}} \\
& \qquad +c\left[(\alpha+\beta) - r^-\right]\max\{\lambda,\mu\}\|(u,v)\|_E^{r^+}
\end{aligned}
$$

Thus we get

$$
\|(u,v)\|_E \leqslant \left[\frac{c(\alpha+\beta-r^-)max\{\lambda,\mu\}}{\alpha+\beta-\max\{p^+,q^+\}}\right]^{\frac{1}{\max\{p^+,q^+\}-r^+}} \tag{3.4}
$$

If λ and μ are sufficiently small, we find $\|(u,v)\| \leqslant 1$, which contradicts the fact that $\|(u,v)\| > 2$. Hence, we conclude that $\mathcal{N}^0_{\lambda,\mu} = \emptyset$. □

From Lemma 3.2 there exists $C_{\lambda,\mu}$ such that for $0 < \max\{\lambda,\mu\} \leqslant C_{\lambda,\mu}$ we can write

$$
\mathcal{N}_{\lambda,\mu}(\Omega) = \mathcal{N}^+_{\lambda,\mu}(\Omega) \cup \mathcal{N}^-_{\lambda,\mu}(\Omega)
$$

Therefore, we can put

$$
\gamma^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}(\Omega)} J_{\lambda,\mu}(u,v) \quad \textit{and} \quad \gamma^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)} J_{\lambda,\mu}(u,v)
$$

Lemma 3.3 *if $0 < \max\{\lambda,\mu\} \leqslant C_{\lambda,\mu}$, then for all $(u,v) \in \mathcal{N}^+_{\lambda,\mu}(\Omega)$ we have $J_{\lambda,\mu}(u,v) < 0$*

Proof: Let $(u,v) \in \mathcal{N}^+_{\lambda,\mu}(\Omega)$. By the definition of $J_{\lambda,\mu}$ we can write

$$
\begin{aligned}
J_{\lambda,\mu}(u,v) \leqslant & \ \tfrac{1}{p^-}\left(\frac{1}{2}\int_Q \tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u|^{\bar{p}(x)}dx\right) \\
& + \tfrac{1}{q^-}\left(\frac{1}{2}\int_Q \tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v|^{\bar{q}(x)}dx\right) \\
& - \tfrac{1}{r^+}\int_\Omega(\lambda|u|^{r(x)} + \mu|v|^{r(x)})dx - \tfrac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dx
\end{aligned}
$$

Since $(u,v)\in\mathcal{N}^+_{\lambda,\mu}(\Omega)$ we have

$$p^+\left(\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{2|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega |u|^{\bar{p}(x)}dx\right)+q^+\left(\int_Q \frac{|v(x)-v(y)|^{q(x,y)}}{2|x-y|^{N+sq(x,y)}}dxdy\right.$$
$$\left.+\int_\Omega |v|^{\bar{q}(x)}dx\right)-\lambda r^-\int_\Omega |u|^{r(x)}dx-\mu r^-\int_\Omega |v|^{r(x)}dx-2(\alpha+\beta)\int_\Omega |u|^\alpha|v|^\beta dx>0 \tag{3.5}$$

Now, if we multiply (3.1) by $(-r^-)$ and add with (3.5), we get

$$\left(\max\{p^+,q^+\}-r^-\right)\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)+2[r^--(\alpha+\beta)]\int_\Omega |u|^\alpha|v|^\beta dx>0 \tag{3.6}$$

So,

$$\int_\Omega |u|^\alpha|v|^\beta dx<\frac{\max\{p^+,q^+\}-r^-}{2\left(\alpha+\beta-r^-\right)}\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right). \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}J_{\lambda,\mu}(u,v)\ &\leqslant \max\{\tfrac{1}{p^-},\tfrac{1}{q^-}\}\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)\\ &-\tfrac{1}{r^+}\left[\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)-2\int_\Omega |u|^\alpha|v|^\beta dx\right]-\tfrac{2}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dx\\ &\leqslant\left(\max\{\tfrac{1}{p^-},\tfrac{1}{q^-}\}-\tfrac{1}{r^+}\right)\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)+\left(\tfrac{2}{r^+}-\tfrac{2}{\alpha+\beta}\right)\int_\Omega |u|^\alpha|v|^\beta dx\end{aligned} \tag{3.8}$$

By applying (3.7), and taking to account (1.3), it follows that

$$\begin{aligned}J_{\lambda,\mu}(u,v)\ &\leqslant\left(\max\{\tfrac{1}{p^-},\tfrac{1}{q^-}\}-\tfrac{1}{r^+}\right)\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)\\ &+\left(\tfrac{2}{r^+}-\tfrac{2}{\alpha+\beta}\right)\tfrac{\max\{p^+,q^+\}-r^-}{2\left(\alpha+\beta-r^-\right)}\left(\rho_{s,p(x,y),Q}(u)+\rho_{s,q(x,y),Q}(v)\right)\\ &\leqslant\left[\tfrac{r^+-\min\{p^-,q^-\}}{r^+\min\{p^-,q^-\}}+\tfrac{\max\{p^+,q^+\}-r^-}{r^+(\alpha+\beta)}\right]\|(u,v)\|_E^{\max\{\delta_p,\delta_q\}}\\ &\leqslant\tfrac{(\alpha+\beta)(r^+-\min\{p^-,q^-\})+\min\{p^-,q^-\}(\max\{p^+,q^+\}-r^-)}{r^+\min\{p^-,q^-\}(\alpha+\beta)}\|(u,v)\|_E^{\max\{\delta_p,\delta_q\}}<0.\end{aligned}$$

where

$$\delta_p=\begin{cases}p^+ & if\quad \|u\|_{X^{p(x,y)}}\geqslant 1\\ p^- & if\quad \|u\|_{X^{p(x,y)}}\leqslant 1\end{cases}\qquad\qquad \delta_q=\begin{cases}q^+ & if\quad \|v\|_{X^{q(x,y)}}\geqslant 1\\ q^- & if\quad \|v\|_{X^{q(x,y)}}\leqslant 1\end{cases}$$

Hence, we have

$$\gamma^+_{\lambda,\mu}=\inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}(\Omega)}J_{\lambda,\mu}(u,v)<0.$$

□

Proposition 3.1 *If $0<\max\{\lambda,\mu\}<C_{\lambda,\mu}$, there exists a minimizer of $J_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$.*

Proof: Since $J_{\lambda,\mu}$ is bounded on $\mathcal{N}_{\lambda,\mu}(\Omega)$ and so on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$, then there exists a minimizing sequence $\{u_n^+,v_n^+\}\subset\mathcal{N}^+_{\lambda,\mu}(\Omega)$ such that

$$\lim_{n\to+\infty}J_{\lambda,\mu}(u_n^+,v_n^+)=\inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}(\Omega)}J_{\lambda,\mu}(u,v)<0. \tag{3.9}$$

Since $J_{\lambda,\mu}$ is coercive, u_n^+ is bounded in $X^{p(x,y)}(\Omega)$ and v_n^+ is bounded in $X^{q(x,y)}(\Omega)$
So, for a subsequence we have $u_n^+\rightharpoonup u_0^+$ in $X^{q(x,y)}(\Omega)$ and $v_n^+\rightharpoonup v_0^+$ in $X^{q(x,y)}(\Omega)$

by the compact embedding we have

$$\begin{cases}u_n^+\rightharpoonup u_0^+ & in\quad L^{r(x)}(\Omega)\\ u_n^+\rightharpoonup u_0^+ & in\quad L^{r(x)}(\Omega)\end{cases}$$

and

$$\begin{cases} u_n^+ \rightharpoonup u_0^+ & in \quad L^{\alpha+\beta}(\Omega) \\ u_n^+ \rightharpoonup u_0^+ & in \quad L^{\alpha+\beta}(\Omega) \end{cases}$$

then $u_n^+ \to u_0^+$ a.e. in $\mathbb{R}^N$ and $v_n^+ \to v_0^+$ a.e. in $\mathbb{R}^N$ and there exists $l_1,\ l_2 \in L^{r(x)}(\mathbb{R}^N)$ such that

$$|u_n^+| \leqslant l_1 \quad and \quad |v_n^+| \leqslant l_2$$

Then

$$\int_\Omega (\lambda|u_n^+|^{r(x)} + \mu|v_n^+|^{r(x)})dx \to \int_\Omega (\lambda|u_0^+|^{r(x)} + \mu|v_0^+|^{r(x)})dx \tag{3.10}$$

Next we show that $(u_n^+, v_n^+) \to (u_0^+, v_0^+)$ in E. If not, then

$$\|(u,v)\|_E < \liminf_{n\to+\infty} \|(u_n^+, v_n^+)\|_E$$

Using the fact that $(u_n^+, v_n^+) \in \mathcal{N}_{\lambda,\mu}^+ \subset \mathcal{N}_{\lambda,\mu}$ we get

$$\begin{aligned} J_{\lambda,\mu}(u_n^+, v_n^+) \quad &\geqslant \left(\min\{\tfrac{1}{p^+}, \tfrac{1}{q^+}\} - \tfrac{1}{\alpha+\beta} \right) \|(u_n^+, v_n^+)\|_E^{\min\{\delta_p,\delta_q\}} \\ &+ \left(\tfrac{1}{\alpha+\beta} - \tfrac{1}{r^-} \right) \int_\Omega (\lambda|u_n^+|^{r(x)} + \mu|v_n^+|^{r(x)})dx. \end{aligned} \tag{3.11}$$

Thus

$$\begin{aligned} \gamma_{\lambda,\mu}^+ = \lim_{n\to+\infty} J_{\lambda,\mu}(u_n^+, v_n^+) \quad &\geqslant \left(\min\{\tfrac{1}{p^+}, \tfrac{1}{q^+}\} - \tfrac{1}{\alpha+\beta} \right) \lim_{n\to+\infty} \|(u_n^+, v_n^+)\|_E^{\min\{\delta_p,\delta_q\}} \\ &+ \left(\tfrac{1}{\alpha+\beta} - \tfrac{1}{r^-} \right) \lim_{n\to+\infty} \int_\Omega (\lambda|u_n^+|^{r(x)} + \mu|v_n^+|^{r(x)})dx. \end{aligned} \tag{3.12}$$

That is

$$\begin{aligned} \gamma_{\lambda,\mu}^+ = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v) \quad &\geqslant \left(\min\{\tfrac{1}{p^+}, \tfrac{1}{q^+}\} - \tfrac{1}{\alpha+\beta} \right) \|(u_0^+, v_0^+)\|_E^{\min\{\delta_p,\delta_q\}} \\ &+ \left(\tfrac{1}{\alpha+\beta} - \tfrac{1}{r^-} \right) \int_\Omega (\lambda|u_0^+|^{r(x)} + \mu|v_0^+|^{r(x)})dx. \end{aligned} \tag{3.13}$$

For $\|u\|_{X^{p(x,y)}} > 1$ and $\|v\|_{X^{q(x,y)}} > 1$ we have

$$\gamma_{\lambda,\mu}^+ = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v) > 0 \tag{3.14}$$

However, for any $(u,v) \in \mathcal{N}_{\lambda,\mu}^+(\Omega)$ we have $J_{\lambda,\mu}(u,v) < 0$. So this is a contradiction. Hence

$$(u_n^+, v_n^+) \to (u_0^+, v_0^+) \quad in \quad E$$

and

$$J_{\lambda,\mu}(u_0^+, v_0^+) = \lim_{n\to+\infty} J_{\lambda,\mu}(u_n^+, v_n^+) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v)$$

Namely, (u_0^+, v_0^+) is a minimizer of $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}^+(\Omega)$, then it is a critical point of $J_{\lambda,\mu}$. □

Lemma 3.4 *if* $0 < \max\{\lambda,\mu\} \leqslant C_{\lambda,\mu}$, *then for all* $(u,v) \in \mathcal{N}_{\lambda,\mu}^-(\Omega)$ *we have* $J_{\lambda,\mu}(u,v) > 0$

Proof: Let $(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)$. By the definition of $J_{\lambda,\mu}$ we can write

$$\begin{aligned} J_{\lambda,\mu}(u,v) &\geqslant \min\{\tfrac{1}{p^+},\tfrac{1}{q^+}\}\|(u,v)\|_E^{\min\{\delta_p,\delta_q\}} - \tfrac{1}{r^-}\int_\Omega(\lambda|u|^{r(x)}+\mu|v|^{r(x)})dx \\ &\quad -\tfrac{1}{\alpha+\beta}\int_\Omega|u|^\alpha|v|^\beta dx \\ &\geqslant \left(\min\{\tfrac{1}{p^+},\tfrac{1}{q^+}\}-\tfrac{1}{\alpha+\beta}\right)\|(u,v)\|_E^{\min\{\delta_p,\delta_q\}} \\ &\quad +C\left(\tfrac{1}{\alpha+\beta}-\tfrac{1}{r^-}\right)\max\{\lambda,\mu\}\|(u,v)\|_E^{r^+} \end{aligned}$$

If we chose $\max\{\lambda,\mu\}<\dfrac{r^-\left(\alpha+\beta-\max\{p^+,q^+\}\right)}{\max\{p^+,q^+\}\left(\alpha+\beta-r^-\right)}$, we get $J_{\lambda,\mu}(u,v)>0$. Moreover, since $\mathcal{N}_{\lambda,\mu}(\Omega)=\mathcal{N}^+_{\lambda,\mu}(\Omega)\cup\mathcal{N}^-_{\lambda,\mu}(\Omega)$ and $\mathcal{N}^+_{\lambda,\mu}(\Omega)\cap\mathcal{N}^-_{\lambda,\mu}(\Omega)=\emptyset$, from Lemma 3.3 we must have $(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)$. □

Proposition 3.2 *If $0<\max\{\lambda,\mu\}<C_{\lambda,\mu}$, there exists a minimizer of $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}(\Omega)$.*

Proof: Since $J_{\lambda,\mu}$ is bounded on $\mathcal{N}_{\lambda,\mu}(\Omega)$ and so on $\mathcal{N}^-_{\lambda,\mu}(\Omega)$, there exists a minimizing sequence $\{u_n^-,v_n^-\}\subset\mathcal{N}^-_{\lambda,\mu}(\Omega)$ such that

$$\lim_{n\to+\infty}J_{\lambda,\mu}(u_n^-,v_n^-)=\inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)}J_{\lambda,\mu}(u,v)>0 \tag{3.15}$$

Since $J_{\lambda,\mu}$ is coercive, $\{u_n^-,v_n^-\}$ is bounded in E. Then, for a subsequence, we have $u_n^-\rightharpoonup u_0^-$ in $X^{p(x,y)}$ and $v_n^-\rightharpoonup v_0^-$ in $X^{p(x,y)}$.
By the compact embedding we have

$$u_n^-\longrightarrow u_0^- \quad and \quad v_n^-\longrightarrow v_0^- \quad in \quad L^{r(x)}(\Omega)$$

and

$$u_n^-\longrightarrow u_0^- \quad and \quad v_n^-\longrightarrow v_0^- \quad in \quad L^{\alpha+\beta}(\Omega)$$

Moreover, if $(u_0^-,v_0^-)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)$, then there exists a constant $t>0$ such that $(tu_0^-,tv_0^-)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)$. Indeed, since

$$\begin{aligned} < I'_{\lambda,\mu}(u,v),(u,v)> &= \frac{1}{2}\int_Q p(x,y)\tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega\bar{p}(x)|u|^{\bar{p}(x)}dx \\ &\quad +\frac{1}{2}\int_Q q(x,y)\tfrac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega\bar{q}(x)|v|^{\bar{q}(x)}dx \\ &\quad -\int_\Omega r(x)\left(\lambda|u|^{r(x)}+\mu|v|^{r(x)}\right)dx-2(\alpha+\beta)\int_\Omega|u|^\alpha|v|^\beta dx \end{aligned}$$

Then

$$\begin{aligned} < I'_{\lambda,\mu}(tu_0^-,tv_0^-),(tu_0^-,tv_0^-)> &\leqslant t^{p^+}p^+\left(\frac{1}{2}\int_Q\tfrac{|u_0^-(x)-u_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega|u_0^-(x)|^{\bar{p}(x)}dx\right) \\ &\quad +t^{q^+}q^+\left(\frac{1}{2}\int_Q\tfrac{|v_0^-(x)-v_0^-(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v_0^-(x)|^{\bar{q}(x)}dx\right) \\ &\quad -t^{r^-}r^-\int_\Omega(\lambda|u_0^-|^{r(x)}+\mu|v_0^-|^{r(x)})dx \\ &\quad -2(\alpha+\beta)t^{\alpha+\beta}\int_\Omega|u_0^-|^\alpha|v_0^-|^\beta dx \\ &\leqslant t^{p^++q^+}\max\{p^+,q^+\}\|(u_0^-,v_0^-)\|_E^{max\{\gamma_p,\gamma_q\}} \\ &\quad -t^{r^-}r^-\int_\Omega(\lambda|u_0^-|^{r(x)}+\mu|v_0^-|^{r(x)})dx \\ &\quad -2(\alpha+\beta)t^{\alpha+\beta}\int_\Omega|u_0^-|^\alpha|v_0^-|^\beta dx \end{aligned}$$

Since $r^- < \max\{p^+, q^+\} < \alpha+\beta$, then it follows

$$< I'_{\lambda,\mu}(tu_0^-, v_0^-), (tu_0^-, v_0^-) >< 0$$

So, by the definition of $\mathcal{N}^-_{\lambda,\mu}(\Omega)$, we get $(tu_0^-, tv_0^-) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$

Now, we shall show that $(u_n^-, v_n^-) \longrightarrow (tu_0^-, v_0^-)$ in E. Otherwise, suppose $(u_n^-, v_n^-) \nrightarrow (tu_0^-, v_0^-)$ in E. Using the fact that

$$\|(u,v)\|_E < \liminf_{n\to+\infty} \|(u_n^-, v_n^-)\|_E$$

we have

$$\begin{aligned} J_{\lambda,\mu}(tu_0^-, tv_0^-) &\leqslant \frac{t^{p^+}}{p^-}\left(\frac{1}{2}\int_Q \frac{|u_0^-(x)-u_0^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_0^-|^{\bar{p}(x)}dx\right) \\ &+\frac{t^{q^+}}{q^-}\left(\frac{1}{2}\int_Q \frac{|v_0^-(x)-v_0^-(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_0^-|^{\bar{q}(x)}dx\right) \\ &-\frac{t^{r^-}}{r^+}\int_\Omega(\lambda|u_0^-|^{r(x)} + \mu|v_0^-|^{r(x)})dx - \frac{t^{\alpha+\beta}}{(\alpha+\beta)}\int_\Omega |u_0^-|^\alpha |v_0^-|^\beta dx \\ &\leqslant \lim_{n\to+\infty}\left(\frac{t^{p^+}}{p^-}\left(\frac{1}{2}\int_Q \frac{|u_n^-(x)-u_n^-(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n^-|^{\bar{p}(x)}dx\right)\right. \\ &+\frac{t^{q^+}}{q^-}\left(\frac{1}{2}\int_Q \frac{|v_n^-(x)-v_n^-(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_n^-|^{\bar{q}(x)}dx\right) \\ &\left.-\frac{t^{r^-}}{r^+}\int_\Omega(\lambda|u_n^-|^{r(x)} + \mu|v_n^-|^{r(x)})dx - \frac{t^{\alpha+\beta}}{(\alpha+\beta)}\int_\Omega |u_n^-|^\alpha |v_n^-|^\beta dx\right) \\ &\leqslant \lim_{n\to+\infty} J_{\lambda,\mu}(tu_n^-, tv_n^-) \leqslant \lim_{n\to+\infty} J_{\lambda,\mu}(u_n^-, v_n^-) \\ &= \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)} J_{\lambda,\mu}(u,v) = \gamma^-_{\lambda,\mu} \end{aligned}$$

This implies that $J_{\lambda,\mu}(tu_0^-, tv_0^-) < \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)} J_{\lambda,\mu}(u,v) = \gamma^-_{\lambda,\mu}$, which is a contradiction.

So, $(u_n^-, v_n^-) \longrightarrow (u_0^-, v_0^-)$ in E, and then

$$J_{\lambda,\mu}(u_0^-, v_0^-) = \lim_{n\to+\infty} J_{\lambda,\mu}(u_n^-, v_n^-) = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)} J_{\lambda,\mu}(u,v) \tag{3.16}$$

Thus, (u_0^-, v_0^-) is a minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}(\Omega)$, then it is a critical point of $J_{\lambda,\mu}$. □

Proof of Theorem 1.1. By Proposition 3.1 and Proposition 3.2 we conclude that $(u_0^-, v_0^-) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$ and $(u_0^+, v_0^+) \in \mathcal{N}^+_{\lambda,\mu}(\Omega)$ are two non trivial solutions of system (1.1). Since $\mathcal{N}^-_{\lambda,\mu}(\Omega) \cap \mathcal{N}^+_{\lambda,\mu}(\Omega) = \emptyset$, then those two solutions are distinct.

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Athmane Boumazourh,
Mathematical Analysis, Algebra and Applications Laboratory,
Faculty of Sciences Aïn Chock,
Hassan II University, Casablanca,
Morocco.
E-mail address: athmane.boumazourh@gmail.com

and

Houria El-Yahyaoui,
Modeling, Applied Mathematics and Intelligent Systems Laboratory,
Faculty of Sciences Dhar El Mahraz,
Sidi Mohamed Ben Abdellah University, Fez,
Morocco.
E-mail address: houria.elyahyaoui@usmba.ac.ma

and

Mohammed Shimi,
Mathematics and Applications to Engineering Sciences Laboratory,
Department of Mathematics and Computer Science, ENS Fez
Sidi Mohamed Ben Abdellah University,
Morocco.
E-mail address: mohammed.shimi2@usmba.ac.ma