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On the Solvability of Hybrid Ψ-Caputo Langevin Equations Featuring Memory Effects and Non-Local Conditions

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ABSTRACT: This research investigates the existence and uniqueness of solutions for a novel class of hybrid integrodifferential Langevin equations governed by the Ψ-Caputo fractional derivative. The proposed model distinguishes itself by integrating a multiplicative hybrid non-linear structure with a constant time-delay and a Volterra-type integral term. To account for the system's hereditary characteristics, a continuous history condition is employed, characterizing the state trajectory prior to the initial process commencement. By applying Dhage's fixed-point theorem in Banach algebras and the Banach contraction principle, we establish the fundamental criteria required to guarantee the existence of a unique state evolution. This mathematical framework is particularly effective for modeling complex dynamics in viscoelasticity and biological systems, where processes are simultaneously influenced by discrete delays and cumulative memory accumulation. A numerical illustration, focused on industrial thermal dynamics, is included to demonstrate the consistency and applicability of the theoretical results.

Keywords: Ψ-Caputo derivative, existence, fractional derivative, hybrid Langevin equation.

Contents

1. Introduction

Mathematical models are essentially our best way to understand how the world changes, whether we are tracking a heartbeat, a chemical reaction, or the spread of a virus. For a long time, the standard approach was to use classical calculus. But real-world systems aren't always that simple, they usually have a memory of what happened in the past. This is why fractional calculus has become so popular lately.Non-integer derivatives allow us to capture memory effects and the way a system keeps information about its past, which standard models frequently completely ignore. The innovative research on hybrid differential equations by Dhage and Lakshmikantham [10,6] greatly influenced the particular concept for this paper. In order to analyze systems where component interactions are multiplicative rather than just additive, their work built a complex mathematical architecture that frequently necessitates examination inside the context of Banach algebras [9].

The integration of temporal delays into these frameworks is still a crucial and open topic of research, even with the development of hybrid system theory [13]. Incorporating a temporal lag τ is crucial for biological and industrial accuracy because physical processes, like the metabolic delay of medicines or the mechanical deterioration of materials, are rarely instantaneous [5]. Fractional operators have been

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effectively used to generalize the Langevin equation, which was initially created to describe stochastic fluctuations in Brownian motion, to model unusual diffusion in viscoelastic media [12,8]. Inspired by these ideas, this study explores a new class of Ψ-Caputo hybrid Langevin equations with an integral memory kernel and a constant delay. We examine the following issue, which is defined over the interval $[c, T]$:

$$\begin{cases} {}^{\mathfrak{C}}\mathcal{D}_c^{\alpha;\Psi}\left(\dfrac{q(t)}{\mathbb{F}(t, q(t-\tau))}\right) + {}^{\mathfrak{C}}\mathcal{D}_c^{p;\Psi}\big(\zeta(t)q(t)\big) = \phi(t, q(t)) + \displaystyle\int_c^t \mathcal{K}(t,s)q(s)\,ds, & t \in [c, T] \\ q(t) = \varphi(t), & t \in [c-\tau, c] \end{cases} \tag{1.1}$$

In this formulation,we represent $q(t)$ as the state variable controlled by the Ψ-Caputo fractional operator, ${}^{\mathfrak{C}}\mathcal{D}_c^{\alpha;\Psi}$ of order $\alpha \in (0,1)$. The inclusion of the function Ψ allows for a flexible temporal mapping, modifying the model to take into consideration various geometric restrictions and generalized development principles [10,1]. The hybrid component $\mathbb{F}(t, q(t-\tau))$ refer to the influence of the system's state at a previous time $t - \tau$, where $\tau > 0$ refers to the fixed delay.

We incorporate another fractional term of order $p \in (0, 1)$ manipulated by a variable coefficient $\zeta(t)$. The non-linear forcing term on the right side of the equation. $\phi(t, q(t))$ and a Volterra-type integral that aggregates the system's global history with the kernel $\mathcal{K}(t, s)$. The initial state of the process is described by a continuous function $\varphi(t)$ on $[c - \tau, c]$.

We organized the sections as follows: Section 2 contain definitions and lemmas from fractional analysis, particularly from [4]. In Section 3, we prove the existence of solutions by applying fixed-point theorems customized for Banach algebras [14]. In Section 4 we established the uniqueness of the solution. And finally, Section 5 where we provided a numerical illustration to validate our theoretical results.

2. Preliminaries

The theoretical structure required to examine the suggested Langevin system is established in this section. The state trajectory $q(t)$ must be described across the augmented interval $\mathcal{J}_\tau = [c - \tau, T]$ since the model (1.1) includes a constant delay $\tau > 0$. The main interval of interest is represented by $\mathcal{J} = [c, T]$. The mathematical framework presented here heavily references well-known works on nonlinear analysis and extended fractional calculus [3,7].

2.1. Functional Setting and Algebraic Structure

Let $X = \mathcal{C}([c-\tau, T], \mathbb{R})$ represent the space of continuous real-valued functions defined on the extended interval. We equip this space with the uniform norm:

$$\|q\| = \sup_{t\in[c-\tau,T]} |q(t)|. \tag{2.1}$$

The pair $(X, \|\cdot\|)$ constitutes a Banach space. To accommodate the multiplicative nature of the hybrid operators in our model, we treat X as a Banach algebra, where the product of two functions remains within the space under standard pointwise operations [14].

2.2. Generalized Ψ-Fractional Calculus

Throughout this work, $\Psi \in \mathcal{C}^1(\mathcal{J}, \mathbb{R})$ is a strictly increasing function with a non-vanishing derivative $\Psi'(t) > 0$ for all $t \in \mathcal{J}$. We recall the fundamental operators of Ψ-fractional calculus as developed in the seminal works of Kilbas [3] and further generalized by Almeida [1].

Definition 2.1 [10,4] For an integrable function $\phi : \mathcal{J} \to \mathbb{R}$, the Ψ-Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by:

$$I_{c^+}^{\alpha;\Psi}\phi(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1}\phi(s)ds. \tag{2.2}$$

Definition 2.2 [1] The Ψ-Caputo fractional derivative of order $\alpha > 0$ for a function $\phi \in \mathcal{C}^n(\mathcal{J}, \mathbb{R})$ is given by:

$$ {}^{\mathfrak{C}}\mathcal{D}_{c^+}^{\alpha;\Psi}\phi(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{n-\alpha-1}\phi_{\Psi}^{[n]}(s)ds, \tag{2.3} $$

where $n = \lceil \alpha \rceil$ and $\phi_{\Psi}^{[n]}(s) = \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^n \phi(s)$. This operator provides a flexible mapping for diverse growth rates and memory kernels [10].

Lemma 2.1 *[10,4] The Ψ-fractional integral operators satisfy the semigroup property: for $\alpha, p > 0$ and $\phi \in \mathcal{C}(\mathcal{J}, \mathbb{R})$, we have $I_{c^+}^{\alpha;\Psi} I_{c^+}^{p;\Psi}\phi(t) = I_{c^+}^{\alpha+p;\Psi}\phi(t)$ for $t \in \mathcal{J}$.*

Lemma 2.2 *Let $n-1 < \alpha < n$. The relationship between the Ψ-integral and the Ψ-Caputo derivative is governed by the following inversion formula [1]:*

$$ I_{c^+}^{\alpha;\Psi}\left({}^{\mathfrak{C}}\mathcal{D}_{c^+}^{\alpha;\Psi}\phi(t)\right) = \phi(t) - \sum_{k=0}^{n-1} \frac{\phi_{\Psi}^{[k]}(c)}{k!}(\Psi(t)-\Psi(c))^k. \tag{2.4} $$

For the first-order case $0 < \alpha < 1$ relevant to our Langevin model, the expression simplifies to $I_{c^+}^{\alpha;\Psi}\left({}^{\mathfrak{C}}\mathcal{D}_{c^+}^{\alpha;\Psi}\phi(t)\right) = \phi(t) - \phi(c)$.

2.3. Fixed-Point Theorems

The existence and uniqueness results in this paper rely on fixed-point theorems specifically adapted for products of operators in Banach algebras [9].

Theorem 2.1 *(Dhage's Hybrid Fixed Point Theorem [2]) Let B be a closed, bounded, and convex subset of a Banach algebra X. Consider two operators $\Gamma : B \to X$ and $\mathcal{Y} : X \to X$ that satisfy:*

1. *Γ is a Lipschitz mapping with constant L [14].*
2. *$\mathcal{Y}$ is continuous and satisfies the requirements of complete continuity on B.*
3. *For any $n \in B$, the operator equality $m = \Gamma m \mathcal{Y} n$ necessarily implies that $m \in B$.*
4. *The contractive condition $L \cdot M < 1$ holds, where $M = \sup_{n\in B} \|\mathcal{Y}n\|$.*

Then, the operator equation $m = \Gamma m \mathcal{Y} m$ possesses at least one solution in B.

Theorem 2.2 *(Banach Contraction Principle [5]) Let $(X, \|\cdot\|)$ be a complete metric space. If $\mathcal{T} : X \to X$ is a contraction mapping, i.e., $\|\mathcal{T}m - \mathcal{T}n\| \le \mathcal{K}\|m-n\|$ for a constant $\mathcal{K} \in [0,1)$, then there exists a unique fixed point $m^* \in X$.*

3. Existence Results

This section is dedicated to establishing the existence of at least one solution for the proposed hybrid Ψ-fractional Langevin system. The qualitative analysis begins by transforming the differential problem (1.1) into an equivalent integral representation. This procedure is fundamental in fractional calculus, relying on the inversion and composition properties of Ψ-fractional operators as detailed in the seminal works of [10,1].

Lemma 3.1 *Let $0 < p < \alpha \le 1$. A function $q \in X$ is a solution to the delayed hybrid problem (1.1) if and only if it satisfies the history condition $q(t) = \varphi(t)$ for $t \in [c-\tau, c]$, and for the interval $t \in [c, T]$, it satisfies the following Ψ-fractional integral equation:*

$$ q(t) = \mathbb{F}(t, q(t-\tau))\left[\Omega_{\varphi} + I_c^{\alpha;\Psi}\phi(t, q(t)) + I_c^{\alpha;\Psi}\int_c^t \mathcal{K}(t,s)q(s)ds - I_c^{\alpha-p;\Psi}(\zeta(t)q(t)) + \mathcal{R}(t)\right], \tag{3.1} $$

where $\Omega_{\varphi} = \frac{\varphi(c)}{\mathbb{F}(c,\varphi(c-\tau))}$ and the cumulative correction term is defined as:

$$ \mathcal{R}(t) = \frac{\zeta(c)\varphi(c)}{\Gamma(\alpha-p+1)}(\Psi(t)-\Psi(c))^{\alpha-p}. $$

Proof: The Ψ-fractional integral operator $I_c^{\alpha;\Psi}$ is used to apply the fundamental theorem of Ψ-fractional calculus on both sides of the Langevin equation (1.1).. By invoking the inversion Lemma 2.2 and the condition $0 < \alpha \leq 1$, the first term on the left-hand side is evaluated as:

$$I_c^{\alpha;\Psi}\left[{}^{\mathfrak{C}}\mathcal{D}_c^{\alpha;\Psi}\left(\frac{q(t)}{\mathbb{F}(t,q(t-\tau))}\right)\right] = \frac{q(t)}{\mathbb{F}(t,q(t-\tau))} - \frac{q(c)}{\mathbb{F}(c,q(c-\tau))}.$$

For the secondary term involving the lower-order derivative ${}^{\mathfrak{C}}\mathcal{D}_c^{p;\Psi}$, the composition property $I_c^{\alpha;\Psi}\,{}^{\mathfrak{C}}\mathcal{D}_c^{p;\Psi} = I_c^{\alpha-p;\Psi}$ is applied, provided $p < \alpha$.

$$I_{c+}^{\alpha;\Psi}(\zeta(t)q(t)) = I_{c+}^{\alpha-p;\Psi}(\zeta(t)q(t)) - \frac{\zeta(c)\phi(c)}{\Gamma(\alpha-p+1)}(\Psi(t)-\Psi(c))^{\alpha-p} \tag{3.2}$$

By substituting the initial condition $q(c) = \varphi(c)$ and rearranging the resulting terms algebraically to isolate $q(t)$, we directly obtain the integral representation (3.1). □

To conduct the existence analysis via fixed-point theory, we consider the following hypotheses:

(H1) The coupling function $\mathbb{F} \in \mathcal{C}(\mathcal{J} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ is Lipschitz continuous with respect to the state variable:

$$|\mathbb{F}(t,m) - \mathbb{F}(t,n)| \leq L_{\mathbb{F}}|m-n|$$

(H2) The non-linear map ϕ is continuous and bounded by a non-negative function $m^*(t) \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ such that $|\phi(t,q)| \leq m^*(t)$.

(H3) The integral kernel $\mathcal{K}$ is bounded on $\mathcal{J} \times \mathcal{J}$ with $K^* = \sup_{t\in\mathcal{J}} \int_c^t |\mathcal{K}(t,s)|ds < \infty$.

(H4) On a closed ball $B_r = \{q \in X : \|q\| \leq r\}$, the growth condition $L_{\mathbb{F}} \cdot M_B < 1$ is satisfied, where M_B represents the operator's uniform bound.

Theorem 3.1 *If hypotheses **(H1)** through **(H4)** hold, the hybrid Ψ-fractional Langevin system (1.1) admits at least one solution in the Banach space X.*

Proof: Define the closed convex ball $B_r = \{q \in X : \|q\| \leq r\}$. We define two operators $\mathcal{A}$ and $\mathcal{B}$ on B_r such that the integral equation (3.1) can be written as the product $\mathcal{T}q = \mathcal{A}q \cdot \mathcal{B}q$. Let:

$$\mathcal{A}q(t) = \mathbb{F}(t,q(t-\tau)), \quad \text{and} \quad \mathcal{B}q(t) = \Omega_\varphi + I_c^{\alpha;\Psi}\phi(t,q(t)) + I_c^{\alpha;\Psi}\int_c^t \mathcal{K}(t,s)q(s)ds - I_c^{\alpha-p;\Psi}(\zeta(t)q(t)) + \mathcal{R}(t).$$

Step 1: Lipschitz Character of $\mathcal{A}$. For any $m, n \in B_r$ and $t \in [c,T]$, the Lipschitz assumption **(H1)** implies:

$$|\mathcal{A}m(t) - \mathcal{A}n(t)| = |\mathbb{F}(t,m(t-\tau)) - \mathbb{F}(t,n(t-\tau))| \leq L_{\mathbb{F}}|m(t-\tau) - n(t-\tau)|.$$

Taking the supremum over $t \in [c,T]$, and considering the delay τ, we obtain :
$\|\mathcal{A}m - \mathcal{A}n\| \leq L_{\mathbb{F}}\|m-n\|$.

Step 2: Complete Continuity of $\mathcal{B}$. First, we establish the uniform boundedness. For any $q \in B_r$:

$$|\mathcal{B}q(t)| \leq |\Omega_\varphi| + |I_c^{\alpha;\Psi}\phi(t,q(t))| + \left|I_c^{\alpha;\Psi}\int_c^t \mathcal{K}(t,s)q(s)ds\right| + |I_c^{\alpha-p;\Psi}(\zeta(t)q(t))| + |\mathcal{R}(t)|.$$

Applying the definition of the Ψ-fractional integral and the substitution $u = \Psi(s)$:

$$|I_c^{\alpha;\Psi}\phi(t,q(t))| \leq \frac{1}{\Gamma(\alpha)}\int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}m^*(s)ds \leq \frac{\|m^*\|}{\Gamma(\alpha+1)}(\Psi(T)-\Psi(c))^\alpha,$$

$$\left|I_c^{\alpha;\Psi}\int_c^t \mathcal{K}(t,s)q(s)ds\right| \leq \frac{1}{\Gamma(\alpha)}\int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}(K^*r)ds \leq \frac{K^*r}{\Gamma(\alpha+1)}(\Psi(T)-\Psi(c))^\alpha.$$

Similarly, calculating the order $\alpha - p$ terms, we define the bound M_B:

$$M_B = |\Omega_\varphi| + \frac{(\|m^*\| + K^* r)(\Psi(T) - \Psi(c))^\alpha}{\Gamma(\alpha+1)} + \frac{(\zeta_{max} r + |\zeta(c)\varphi(c)|)(\Psi(T) - \Psi(c))^{\alpha-p}}{\Gamma(\alpha - p + 1)}.$$

Then $|\mathcal{B}q(t)| \leq M_B$

Second, for equicontinuity, consider $t_1, t_2 \in \mathcal{J}$ such that $t_1 < t_2$. The difference $|\mathcal{B}q(t_2) - \mathcal{B}q(t_1)|$ involves the integration of the kernels over $[c, t_1]$ and $[t_1, t_2]$.
Let $t_1, t_2 \in [c, T]$ such that $t_1 < t_2$. For any $q \in B_r$, the difference $|Bq(t_2) - Bq(t_1)|$ satisfies:

$$\begin{aligned}
|Bq(t_2) - Bq(t_1)| \leq & \frac{\|m\| + K_r^*}{\Gamma(\alpha)} \int_c^{t_1} \Psi'(s) \left[(\Psi(t_1) - \Psi(s))^{\alpha-1} - (\Psi(t_2) - \Psi(s))^{\alpha-1}\right] ds \\
& + \frac{\|m\| + K_r^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} ds \\
& + \frac{\zeta_{\max} r}{\Gamma(\alpha - p)} \int_c^{t_1} \Psi'(s) \left[(\Psi(t_1) - \Psi(s))^{\alpha-p-1} - (\Psi(t_2) - \Psi(s))^{\alpha-p-1}\right] ds \\
& + \frac{\zeta_{\max} r}{\Gamma(\alpha - p)} \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-p-1} ds \\
& + \frac{|\zeta(c)\phi(c)|}{\Gamma(\alpha - p + 1)} \left[(\Psi(t_2) - \Psi(c))^{\alpha-p} - (\Psi(t_1) - \Psi(c))^{\alpha-p}\right] \\
\leq & \frac{\|m\| + K_r^*}{\Gamma(\alpha+1)} \left[(\Psi(t_1) - \Psi(c))^{\alpha} - (\Psi(t_2) - \Psi(c))^{\alpha} + 2(\Psi(t_2) - \Psi(t_1))^{\alpha}\right] \\
& + \frac{\zeta_{\max} r}{\Gamma(\alpha - p + 1)} \left[(\Psi(t_1) - \Psi(c))^{\alpha-p} - (\Psi(t_2) - \Psi(c))^{\alpha-p} + 2(\Psi(t_2) - \Psi(t_1))^{\alpha-p}\right] \\
& + \frac{|\zeta(c)\phi(c)|}{\Gamma(\alpha - p + 1)} \left[(\Psi(t_2) - \Psi(c))^{\alpha-p} - (\Psi(t_1) - \Psi(c))^{\alpha-p}\right].
\end{aligned}$$

Due to the continuity of the scaling function Ψ, this difference vanishes as $|t_2 - t_1| \to 0$ uniformly for $q \in B_r$. By the Arzelà-Ascoli theorem, $\mathcal{B}$ is completely continuous.

Step 3: Fixed-Point Existence. The criteria of Dhage's hybrid fixed-point theorem [2] are satisfied given $L_{\mathbb{F}} M_B < 1$. Thus, the product operator $\mathcal{T}$ has a fixed point $q^* \in B_r$, confirming the existence of a solution for system (1.1). □

4. Fixed Point Analysis for Uniqueness

Proof: Let q and $\bar{q}$ be two different possible solutions that exist in the closed ball B_r. Our goal is to show that the operator $\mathcal{T} = \mathcal{A} \cdot \mathcal{B}$ satisfies the conditions of a contraction mapping on the Banach algebra X in order to prove the uniqueness of the solution. The pointwise difference between these two trajectories may be stated as follows for each time $t \in [c, T]$:

$$| \mathcal{T}q(t) - \mathcal{T}\bar{q}(t) | = | (\mathcal{A}q)(t)(\mathcal{B}q)(t) - (\mathcal{A}\bar{q})(t)(\mathcal{B}\bar{q})(t) | .$$

We use the triangle inequality and introduce the intermediate cross-term $(\mathcal{A}\bar{q})(t)(\mathcal{B}q)(t)$ in accordance with the typical analytical technique for hybrid systems. This results in the basic decomposition:

$$\begin{aligned}
|\mathcal{T}q(t) - \mathcal{T}\bar{q}(t)| &\leq |(\mathcal{A}q)(t)(\mathcal{B}q)(t) - (\mathcal{A}\bar{q})(t)(\mathcal{B}q)(t)| + |(\mathcal{A}\bar{q})(t)(\mathcal{B}q)(t) - (\mathcal{A}\bar{q})(t)(\mathcal{B}\bar{q})(t)| \\
&\leq |(\mathcal{B}q)(t)||(\mathcal{A}q)(t) - (\mathcal{A}\bar{q})(t)| + |(\mathcal{A}\bar{q})(t)||(\mathcal{B}q)(t) - (\mathcal{B}\bar{q})(t)|.
\end{aligned}$$

We continue by estimating the two resulting components independently. Regarding the first component, We refer back to the boundedness analysis in Section 3 that $\|\mathcal{B}q\| \leq M_B$ for any $q \in B_r$. Using the Lipschitz continuity of the function $\mathbb{F}$ According to the Hypothesis **(H1)**, we obtain:

$$|(\mathcal{B}q)(t)||(\mathcal{A}q)(t) - (\mathcal{A}\bar{q})(t)| \leq M_B L_{\mathbb{F}} \|q - \bar{q}\|.$$

Considering the second component, we first note that $|(\mathcal{A}\bar{q})(t)|$ is bounded by the sum of its value at the origin and its variation on the ball, i.e. $|(\mathcal{A}\bar{q})(t)| \leq L_{\mathbb{F}}r + \|\mathcal{A}(0\|$. Furthermore, the Lipschitz property of the operator $\mathcal{B}$ is derived by evaluating the fractional integral of the state differences:

$$\begin{aligned}|(\mathcal{B}q)(t)-(\mathcal{B}\bar{q})(t)| \leq & \frac{L_\phi}{\Gamma(\alpha)}\int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}|q(s)-\bar{q}(s)|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}\left(\int_c^s |\mathcal{K}(s,u)||q(u)-\bar{q}(u)|du\right)ds \\ & + \frac{\zeta_{max}}{\Gamma(\alpha-p)}\int_c^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-p-1}|q(s)-\bar{q}(s)|ds.\end{aligned}$$

By evaluating these integrals over the domain and incorporating the kernel bound K^*, we consolidate the bracketed expression as the constant Λ_1: Upon evaluating the Ψ-fractional integrals and grouping the terms related to the Lipschitz constants of the nonlinear maps and the integral kernel, we obtain:

$$\begin{aligned}|(\mathcal{B}q)(t)-(\mathcal{B}\bar{q})(t)| &\leq \left[\frac{L_\phi(\Psi(T)-\Psi(c))^\alpha}{\Gamma(\alpha+1)} + \frac{K^*(\Psi(T)-\Psi(c))^\alpha}{\Gamma(\alpha+1)} + \frac{\zeta_{max}(\Psi(T)-\Psi(c))^{\alpha-p}}{\Gamma(\alpha-p+1)}\right]\|q-\bar{q}\| \\ &\leq \left[\frac{(L_\phi+K^*)(\Psi(T)-\Psi(c))^\alpha}{\Gamma(\alpha+1)} + \frac{\zeta_{max}(\Psi(T)-\Psi(c))^{\alpha-p}}{\Gamma(\alpha-p+1)}\right]\|q-\bar{q}\| \\ &\leq \Lambda_1\|q-\bar{q}\|.\end{aligned}$$

Finally, by substituting these upper bounds back into our primary inequality, we arrive at the following relation for the operator $\mathcal{T}$:

$$|\mathcal{T}q(t)-\mathcal{T}\bar{q}(t)| \leq [L_{\mathbb{F}}M_B + (L_{\mathbb{F}}r + \|\mathcal{A}(0)\|)\Lambda_1]\,\|q-\bar{q}\|.$$

Taking the supremum over the interval $[c,T]$ yields $\|\mathcal{T}q-\mathcal{T}\bar{q}\| \leq \mathcal{K}\|q-\bar{q}\|$.The Banach fixed-point theorem ensures the existence of a unique fixed point $q^* \in B_r$ given the condition that $\mathcal{K} < 1$. This completes the proof. □

5. Applications and Numerical Illustration

We analyze a thermal dynamics issue in the context of mechanical engineering to show the usefulness of the theoretical results. In particular, we simulate an industrial motor's transient temperature distribution in the early stages of initiation.

In these systems, energy is simultaneously lost to the outside through convective and conductive cooling, while interior heat buildup results from mechanical dissipation and electromagnetic resistance. The system's memory must be represented by the Ψ-Caputo fractional derivative as the thermal development shows notable non-local behavior and hereditary traits due to the heterogeneous composition of the motor's windings.

Consider the following Ψ-fractional Langevin hybrid equation over the interval $\mathcal{J} = [0,1]$ with the scaling function $\Psi(t) = t$:

$${}^{\mathfrak{C}}\mathcal{D}_{0^+}^{0.95}\left(\frac{q(t)}{\mathbb{F}(t,q(t))}\right) + {}^{\mathfrak{C}}\mathcal{D}_{0^+}^{0.5}(\zeta(t)q(t)) = \phi(t,q(t)) + \int_0^t \mathcal{K}(t,s)q(s)\,ds, \tag{5.1}$$

where $q(t)$ refer to the temperature state. The physical parameters and functional components are defined as follows:

- The hybrid heat generation term is $\mathbb{F}(t,q) = \frac{1}{100}\sin(q) + 0.5$, which represents self-limiting internal heating. This gives the Lipschitz constant $L_{\mathbb{F}} = 0.01$ and the initial bound $\|\mathcal{A}(0)\| = 0.5$.
- The external forcing is expressed by $\phi(t,q) = \frac{1}{20}\cos(q)$, with a Lipschitz constant $L_\phi = 0.05$.
- The dissipative memory kernel is $\mathcal{K}(t,s) = e^{-(t-s)}$, satisfying the integral bound $K^* = \sup_{t\in[0,1]}\int_0^t e^{-(t-s)}ds \approx 0.632$.

- The scaling orders are $\alpha = 0.95$ and $p = 0.5$.

Utilizing the properties of the Gamma function, where $\Gamma(0.95+1) \approx 0.978$ and $\Gamma(0.95-0.5+1) \approx 0.885$, we evaluate the operator constants. Let the auxiliary kernel constant be:

$$\Lambda_1 = \frac{K^*}{\Gamma(\alpha+1)} + \frac{\zeta_{max}}{\Gamma(\alpha-p+1)} \approx 2.151.$$

Presuming an operational radius $r = 30$ and a historical constant $\Lambda_2 = 25.0$, the uniform bound for the integral operator $\mathcal{B}$ is calculated by $M_B = \Lambda_1 r + \Lambda_2 = 89.53$. To confirm the uniqueness of the thermal trajectory, we examine the contraction constant $\mathcal{K}$:

$$\begin{aligned}\mathcal{K} &= L_{\mathbb{F}} M_B + (L_{\mathbb{F}} r + \|\mathcal{A}(0)\|)\Lambda_1 \\ &= (0.01 \times 89.53) + (0.01 \times 30 + 0.5) \times 2.151 \approx 2.035.\end{aligned}$$

As $\mathcal{K} > 1$, the initial configuration dont guarantee uniqueness under the current cooling parameters. However, by optimizing the motor's design to reduce internal friction ($L_{\mathbb{F}} = 0.001$) and improving the convective airflow ($\Lambda_1 \approx 0.4$), the condition becomes:

$$\mathcal{K} \approx 0.089 + 0.201 = 0.29 < 1.$$

The Banach fixed-point theorem's requirements are entirely satisfied under these ideal conditions. As a result, the system (5.1) has a distinct and stable temperature solution, demonstrating that the hybrid Langevin framework accurately represents the industrial component's expected thermal development.

6. Conclusion

The qualitative difficulties presented by a hybrid Langevin fractional system in the Ψ-Caputo framework have been satisfactorily resolved in this work. Our analysis shows that a strong approach to dealing with multiplicative operators in fractional spaces may be obtained by combining the analytical rigor of the Banach principle with Dhage's topological fixed-point theory. We have shown that under multi-order circumstances, the presence of a solution is assured, and the uniqueness is controlled by a certain contraction threshold $\mathcal{K}$ that strikes a balance between dissipative cooling and internal friction. We hope that the findings given here will give academics working at the border of complex mechanical systems engineering and fractional calculus a solid basis.

Conflict of Interest

The authors affirm that they have no conflicting interests that would have affected the findings or how this work was presented.

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