



A Novel Framework for Bioperations on e^* -Open Sets: Theoretical Foundations and Share Market Applications

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ABSTRACT: In this paper, we introduce the concept of $e^*_{[\gamma, \gamma']}$ -open sets in topological spaces and examine their properties in detail. Additionally, we propose a new class of functions, termed $(e^*_{[\gamma, \gamma]}, e^*_{[\beta, \beta']})$ -continuous functions, and explore their fundamental characteristics. To demonstrate the practical relevance of these theoretical constructs, we provide an illustrative example showing how $e^*_{[\gamma, \gamma']}$ -open sets can be applied to identify stable financial patterns in the share market. This application bridges abstract topological theory with real-world market dynamics, offering a novel perspective on analyzing stock price movements and trader behavior.

Keywords: bioperation, e^* -open set, $e^*_{[\gamma, \gamma']}$ -open set.

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1. Introduction

The interplay between topology and real-world applications has long fascinated mathematicians and researchers. Njåstad [7] laid the groundwork by introducing α -open sets, which opened new avenues for exploring finer structures in topological spaces. Kasahara [5] further enriched the field by defining operations on topological spaces and introducing α -closed graphs of an operation. Ogata [8] extended this framework by formalizing the γ operation and defining τ_γ , the collection of all γ -open sets in a topological space (X, τ) . H. Z. Ibrahim [8] pushed the boundaries even further by defining operations on $\alpha O(X, \tau)$ and introducing α_γ -open sets, thereby expanding the toolkit for studying topological spaces.

Despite these advancements, there remains a significant gap in applying these abstract mathematical constructs to practical domains such as financial markets. To address this, we focus on e^* -open sets, a concept introduced by E. Ekici [3], who explored their fundamental properties. While Ekici's work provided a robust theoretical foundation, it left open the question of how these sets could be integrated into operational frameworks that mirror real-world scenarios.

In this paper, we bridge this gap by extending the study of e^* -open sets through a novel operation-based framework. Specifically, we introduce $e^*_{[\gamma, \gamma']}$ -open sets, which generalize the notion of e^* -open sets by incorporating two operations, γ and γ' . These operations allow us to model complex interactions and dependencies—qualities that are particularly relevant in dynamic systems like financial markets.

Our work is motivated by the need to identify stable patterns in stock price movements, where trader reactions to price ranges can determine the stability of trading strategies. By leveraging $e^*_{[\gamma, \gamma']}$ -open sets, we provide a mathematical lens to analyze such patterns. For instance, consider a share market with multiple stocks whose price trends form a topology. Conservative and speculative traders react differently to these trends, and their combined reactions can be modeled using γ and γ' operations. A price range

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is deemed stable if it satisfies the conditions of an $e_{[\gamma, \gamma']}^*$ -open set, ensuring that trader reactions remain contained within the intended focus.

This paper makes several key contributions:

We define $e_{[\gamma, \gamma']}^*$ -open sets and explore their properties, providing a comprehensive theoretical framework.

We introduce a new class of functions, $(e_{[\gamma, \gamma']}^*, e_{[\beta, \beta']}^*)$ -continuous functions, and examine their role in preserving structural properties across spaces.

We demonstrate the practical utility of $e_{[\gamma, \gamma']}^*$ -open sets through an application to the share market, showcasing how these sets can identify stable financial patterns.

By integrating abstract topology with real-world applications, our work not only advances the theoretical understanding of e^* -open sets but also highlights their potential to solve practical problems. This dual focus sets the stage for future research at the intersection of mathematics and finance.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let (X, τ) be a topological space and A a subset of X . The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space X is called regular open [10] (resp. δ -open [11], e^* -open [2]) if $A = Int(Cl(A))$ (resp. $A = Int_\delta(A)$, $A \subseteq Cl(Int(Cl_\delta(A)))$). Here, $Int_\delta(A)$ is defined as $Int_\delta(A) = \{x \mid U \in \mathcal{U}(x), Int(Cl(U)) \subseteq A\}$, where $\mathcal{U}(x)$ denotes the set of all open neighborhoods of the point x in X . Equivalently, a subset A of a space X is called regular closed [10] (resp. δ -closed [11], e^* -closed [2]) if $A = Cl(Int(A))$ (resp. $A = Cl_\delta(A)$, $A \supseteq Int(Cl(Int_\delta(A)))$). Here, $Cl_\delta(A)$ is given by $Cl_\delta(A) = \{x \mid \forall U \in \mathcal{U}(x), (Int(Cl(U)) \cap A) \neq \emptyset\}$. The intersection of all e^* -closed sets containing A is called the e^* -closure of A and is denoted by $e^*Cl(A)$. The family of all e^* -open (resp. e^* -closed) sets in a topological space (X, τ) is denoted by $e^*O(X, \tau)$ (resp. $e^*C(X, \tau)$). An operation γ [3] on a topology τ is a mapping from τ into power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V . A subset A of X with an operation γ on τ is called γ -open [6] if for each $x \in A$, there exists an open set U of X containing x such that $U^\gamma \subseteq A$. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ -open sets are called γ -closed. An operation $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ [8] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. A subset A of X is called an α_γ -open set [8] if for each point $x \in A$, there exists an α -open set U of X containing x such that $U^\gamma \subseteq A$. We denote the set of all α_γ -open sets of (X, τ) by $\alpha O(X, \tau)_\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α -regular [8] if for every α -open sets U and V containing $x \in X$, there exists an α -open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α -open [8] if for every α -open set U of $x \in X$, there, exists an α_γ -open set V of X such that $x \in V$ and $V \subseteq U^\gamma$. A non-empty subset A of (X, τ) is said to be $\alpha_{[\gamma, \gamma']}$ -open [4] if $\forall x \in A, \exists \alpha$ -open sets U and V of X containing x , such that $U^\gamma \cap V^{\gamma'} \subseteq A$. The set of all $\alpha_{[\gamma, \gamma']}$ -open sets of (X, τ) is denoted by $\alpha O(X, \tau)_{[\gamma, \gamma']}$. If A_i is $\alpha_{[\gamma, \gamma']}$ -open $\forall i \in I$, then $\bigcup_{i \in I} A_i$ is $\alpha_{[\gamma, \gamma']}$ -open [4].

3. $e_{[\gamma, \gamma']}^*$ -Open Sets

Definition 3.1 A non-empty subset A of (X, τ) is said to be $e_{[\gamma, \gamma']}^*$ -open if $\forall x \in A, \exists e^*$ -open sets U and V of X containing x , such that $U^\gamma \cap V^{\gamma'} \subseteq A$. The set of all $e_{[\gamma, \gamma']}^*$ -open sets of (X, τ) is denoted by $e^*O(X, \tau)_{[\gamma, \gamma']}$.

We suppose that the empty set is $e_{[\gamma, \gamma']}^*$ -open for any operations γ and γ' .

Definition 3.2 An operation γ on $e^*O(X, \tau)$ is said to be e^* -regular if for every e^* -open sets U and V containing $x \in X$, there exists an e^* -open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Proposition 3.1 If A_i is $e_{[\gamma, \gamma']}^*$ -open $\forall i \in I$, then $\bigcup_{i \in I} A_i$ is $e_{[\gamma, \gamma']}^*$ -open.

Proof. Let $x \in \bigcup_{i \in I} A_i$, then $x \in A_i$ for some $i \in I$. Since A_i is an $e^*_{[\gamma, \gamma']}$ -open set, so there exist e^* -open sets U and V of (X, τ) such that $x \in U \cap V \subseteq U^\gamma \cap V^{\gamma'} \subseteq A_i \subseteq \bigcup_{i \in I} A_i$. Therefore $\bigcup_{i \in I} A_i$ is an $e^*_{[\gamma, \gamma']}$ -open set of (X, τ) .

If A and B are two $e^*_{[\gamma, \gamma']}$ -open sets in (X, τ) , then the following example shows that $A \cap B$ need not be $e^*_{[\gamma, \gamma']}$ -open.

Example 3.1 Let $X = \{w_1, w_2, w_3\}$ and $\tau = \{\phi, X, \{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_2, w_3\}\}$ be a topology on X . For each $A \in e^*O(X, \tau)$, we define two operations γ and γ' , by

$$A^\gamma = \begin{cases} Cl(A) & \text{if } w_3 \in A, \\ X & \text{if } w_3 \notin A, \end{cases} \quad \text{and} \quad A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{w_2\}, \\ X & \text{if } A = \{w_2\}. \end{cases}$$

Then, it is obvious that the sets $\{w_1, w_2\}$ and $\{w_2, w_3\}$ are $e^*_{[\gamma, \gamma']}$ -open, however their intersection $\{w_2\}$ is not $e^*_{[\gamma, \gamma']}$ -open.

Proposition 3.2 Let γ and γ' be e^* -regular operations. If A and B are $e^*_{[\gamma, \gamma']}$ -open, then $A \cap B$ is $e^*_{[\gamma, \gamma']}$ -open.

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are $e^*_{[\gamma, \gamma']}$ -open sets, there exist e^* -open sets U, V, W and S such that $x \in U \cap V \subseteq U^\gamma \cap V^{\gamma'} \subseteq A$ and $x \in W \cap S \subseteq W^\gamma \cap S^{\gamma'} \subseteq B$. Since γ and γ' are e^* -regular operations, then there exist an e^* -open sets K and L containing x such that $K^\gamma \cap L^{\gamma'} \subseteq (U^\gamma \cap W^\gamma) \cap (V^{\gamma'} \cap S^{\gamma'}) = (U^\gamma \cap V^{\gamma'}) \cap (W^\gamma \cap S^{\gamma'}) \subseteq A \cap B$. This implies that $A \cap B$ is $e^*_{[\gamma, \gamma']}$ -open set.

Remark 3.1 By the above proposition, if γ and γ' are e^* -regular operations, then $e^*O(X, \tau)_{[\gamma, \gamma']}$ forms a topology on X .

Proposition 3.3 The set A is $e^*_{[\gamma, \gamma']}$ -open in X if and only if for each $x \in A$, there exists an $e^*_{[\gamma, \gamma']}$ -open set B such that $x \in B \subseteq A$.

Proof. Suppose that A is an $e^*_{[\gamma, \gamma']}$ -open set in the space X . Then for each $x \in A$, put $B = A$ which is an $e^*_{[\gamma, \gamma']}$ -open set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists an $e^*_{[\gamma, \gamma']}$ -open set B such that $x \in B \subseteq A$. Thus $A = \bigcup_{x \in A} B_x$, where $B_x \in e^*O(X, \tau)_{[\gamma, \gamma']}$. Therefore, A is an $e^*_{[\gamma, \gamma']}$ -open set.

Proposition 3.4 If A is $e^*_{[\gamma, \gamma']}$ -open, then A is e^* -open.

The converse of the above proposition need not be true in general as it is shown below.

Example 3.2 Let $(X, \tau), \gamma$ and γ' be the same space and the same operations as in Example 3.1 Then $\{w_2\}$ is e^* -open but not $e^*_{[\gamma, \gamma']}$ -open.

Remark 3.2 A subset A is an $e^*_{[id, id']}$ -open set of (X, τ) if and only if A is e^* -open in (X, τ) . The operation $id : e^*O(X, \tau) \rightarrow P(X)$ is defined by $V^{id} = V$ for any set $V \in e^*O(X, \tau)$. This operation is called the identity operation on $e^*O(X, \tau)$. Therefore $e^*O(X, \tau)_{[id, id]} = e^*O(X, \tau)$.

Remark 3.3 From Remark 3.2 we have $e^*O(X, \tau)_{[id, id]} = e^*O(X, \tau) = e^*O(X, \tau)_{id} = e^*O(X, \tau)_{id}$.

Remark 3.4 The following example shows that the concept of $e^*_{[\gamma, \gamma']}$ -open and open are independent.

Example 3.3 Let $X = \{u_1, u_2, u_3\}$ and $\tau = \{\phi, X, \{u_1\}\}$, be a topology on X . For each $A \in e^*O(X, \tau)$ we define two operations γ and γ' , respectively, by $A^\gamma = X$ and $A^{\gamma'} = \begin{cases} A & \text{if } A = \{u_1, u_2\}, \\ X & \text{if } A \neq \{u_1, u_2\}. \end{cases}$

Then, $e^*_{[\gamma, \gamma']}$ -open sets are ϕ, X , and $\{u_1, u_2\}$.

Proposition 3.5 If A is $[\gamma, \gamma']$ -open, then A is $e^*_{[\gamma, \gamma']}$ -open.

The converse of the above proposition need not be true in general as it is shown below.

Example 3.4 Let $X = \{v_1, v_2, v_3\}$ and $\tau = \{\phi, X, \{v_2\}\}$ be a topology on X . For each $A \in e^*O(X, \tau)$, we define two operations γ and γ' , respectively, by $A^\gamma = A^{\gamma'} = A$. Then $\{v_1, v_2\}$ is $e^*_{[\gamma, \gamma']}$ -open but not $[\gamma, \gamma']$ -open.

Proposition 3.6 If A is e^*_γ -open and B is $e^*_{\gamma'}$ -open, then $A \cap B$ is $e^*_{[\gamma, \gamma']}$ -open.

Proposition 3.7 If A is e^*_γ -open, then A is $e^*_{[\gamma, \gamma']}$ -open for any operation γ' .

The converse of Proposition 3.7 need not be true in general as it is shown below.

Example 3.5 Let $X = \{t_1, t_2, t_3\}$ and τ be a discrete topology on X . For each $A \in e^*O(X, \tau)$ we define two operations γ and γ' , respectively, by $A^\gamma = A^{\gamma'} = \begin{cases} A & \text{if } A = \{t_1, t_2\} \text{ or } \{t_2, t_3\}, \\ X & \text{otherwise.} \end{cases}$

Then $\{t_2\}$ is $e^*_{[\gamma, \gamma']}$ -open but not e^*_γ -open.

Proposition 3.8 Let $X : e^*O(X, \tau) \rightarrow P(X)$ be an operation defined by $U^X = X$ for every $U \in e^*O(X, \tau)$. Then A is e^*_γ -open if and only if A is $e^*_{[\gamma, \gamma']}$ -open.

Definition 3.3 A topological space (X, τ) is said to be e^*_γ -regular if for each $x \in X$ and for each e^* -open set V in X containing x , there exists an e^* -open set U in X containing x such that $U^\gamma \subseteq V$.

In the following proposition, we give a condition under which the family of e^* -open sets is identical to the family of e^*_γ -open sets.

Proposition 3.9 A topological space (X, τ) with an operation γ on $e^*O(X, \tau)$ is e^*_γ -regular if and only if $e^*O(X, \tau) = e^*O(X, \tau)_\gamma$.

Remark 3.5 If a topological space (X, τ) is e^*_γ -regular, then $\tau \subseteq e^*O(X, \tau)_\gamma$.

Definition 3.4 A topological space (X, τ) is said to be $e^*_{[\gamma, \gamma']}$ -regular if for each point x in X and every e^* -open set U containing x there exist e^* -open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq U$.

Proposition 3.10 A topological space (X, τ) with operations γ and γ' on $e^*O(X, \tau)$ is $e^*_{[\gamma, \gamma']}$ -regular if and only if $e^*O(X, \tau) = e^*O(X, \tau)_{[\gamma, \gamma']}$.

Proof. Let (X, τ) be $e^*_{[\gamma, \gamma']}$ -regular and $A \in e^*O(X, \tau)$. Since (X, τ) is $e^*_{[\gamma, \gamma']}$ -regular, then for each $x \in A$, there exist e^* -open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq A$. This implies that $A \in e^*O(X, \tau)_{[\gamma, \gamma']}$. But we have $e^*O(X, \tau)_{[\gamma, \gamma']} \subseteq e^*O(X, \tau)$. Therefore $e^*O(X, \tau) = e^*O(X, \tau)_{[\gamma, \gamma']}$.

Conversely, let $e^*O(X, \tau) = e^*O(X, \tau)_{[\gamma, \gamma']}$, $x \in X$ and V be e^* -open of x . Then by assumption V is $e^*_{[\gamma, \gamma']}$ -open set. This implies that there exist e^* -open sets W and S of x such that $W^\gamma \cap S^{\gamma'} \subseteq V$. Therefore (X, τ) is $e^*_{[\gamma, \gamma']}$ -regular.

Proposition 3.11 For e^*_γ -regularity, $e^*_{\gamma'}$ -regularity, and $e^*_{[\gamma, \gamma']}$ -regularity of a space (X, τ) , the following properties hold.

- (1) (X, τ) is $e^*_{[\gamma, X]}$ -regular if and only if it is e^*_γ -regular.
- (2) If (X, τ) is e^*_γ -regular and $e^*_{\gamma'}$ -regular, then it is $e^*_{[\gamma, \gamma']}$ -regular.

Definition 3.5 A subset F of (X, τ) is said to be $e^*_{[\gamma, \gamma']}$ -closed if its complement $X \setminus F$ is $e^*_{[\gamma, \gamma']}$ -open.

We denote the set of all $e^*_{[\gamma, \gamma']}$ -closed sets of (X, τ) by $e^*C(X, \tau)_{[\gamma, \gamma']}$.

Definition 3.6 Let A be a subset of a topological space (X, τ) . The intersection of all $e^*_{[\gamma, \gamma']}$ -closed sets containing A is called the $e^*_{[\gamma, \gamma']}$ -closure of A and denoted by $e^*_{[\gamma, \gamma']}$ -Cl(A).

Proposition 3.12 For a point $x \in X$, $x \in e^*_{[\gamma, \gamma']}$ -Cl(A) if and only if $V \cap A \neq \emptyset$ for every $e^*_{[\gamma, \gamma']}$ -open set V containing x .

Proposition 3.13 Let A and B be subsets of (X, τ) . Then the following hold:

- (1) $A \subseteq e^*_{[\gamma, \gamma']}Cl(A)$.
- (2) If $A \subseteq B$, then $e^*_{[\gamma, \gamma']}Cl(A) \subseteq e^*_{[\gamma, \gamma']}Cl(B)$.
- (3) $A \in e^*C(X, \tau)_{[\gamma, \gamma']}$ if and only if $e^*_{[\gamma, \gamma']}Cl(A) = A$.
- (4) $e^*_{[\gamma, \gamma']}Cl(A) \in e^*C(X, \tau)_{[\gamma, \gamma']}$.
- (5) $e^*_{[\gamma, \gamma']}Cl(A \cap B) \subseteq e^*_{[\gamma, \gamma']}Cl(A) \cap e^*_{[\gamma, \gamma']}Cl(B)$.
- (6) If γ and γ' are e^* -regular, then $e^*_{[\gamma, \gamma']}Cl(A \cup B) = e^*_{[\gamma, \gamma']}Cl(A) \cup e^*_{[\gamma, \gamma']}Cl(B)$.

We introduce the following definition of $e^*Cl_{[\gamma, \gamma']}(A)$.

Definition 3.7 For a subset A of (X, τ) , we define $e^*Cl_{[\gamma, \gamma']}(A)$ as follows: $e^*Cl_{[\gamma, \gamma']}(A) = \{x \in X : (U^\gamma \cap W^{\gamma'}) \cap A \neq \emptyset \text{ holds for every } e^*\text{-open sets } U \text{ and } W \text{ containing } x\}$.

Remark 3.6 In Definitions 3.5, 3.6 and 3.7, put $\gamma' = X$. Then, for any subset A of X , the following hold:

- (1) $e^*_{[\gamma, X]}Cl(A) = e^*_\gamma Cl(A)$.
- (2) $e^*C(X, \tau)_{[\gamma, X]} = \{F : F \text{ is } e^*_\gamma\text{-closed}\}$.
- (3) $e^*Cl_{[\gamma, X]}(A) = e^*Cl_\gamma(A)$.

Proposition 3.14 For a subset A of (X, τ) , we have $A \subseteq e^*Cl(A) \subseteq e^*Cl_{[\gamma, \gamma']}(A) \subseteq e^*_{[\gamma, \gamma']}Cl(A)$.

Theorem 3.1 Let A be a subset of a topological space (X, τ) , the following properties are equivalent:

- (1) $A \in e^*O(X, \tau)_{[\gamma, \gamma']}$.
- (2) $e^*Cl_{[\gamma, \gamma']}(X \setminus A) = X \setminus A$.
- (3) $e^*_{[\gamma, \gamma']}Cl(X \setminus A) = X \setminus A$.
- (4) $X \setminus A \in e^*C(X, \tau)_{[\gamma, \gamma']}$.

Theorem 3.2 For a subset A of (X, τ) , the following properties hold:

- (1) If (X, τ) is $e^*_{[\gamma, \gamma']}$ -regular, then $e^*Cl(A) = e^*Cl_{[\gamma, \gamma']}(A) = e^*_{[\gamma, \gamma']}Cl(A)$.
- (2) $e^*Cl_{[\gamma, \gamma']}(A)$ is an e^* -closed subset of (X, τ) .

Theorem 3.3 Let γ and γ' be e^* -open operations and A a subset of (X, τ) . Then, the following statements hold:

- (1) $e^*Cl_{[\gamma, \gamma']}(A) = e^*_{[\gamma, \gamma']}Cl(A)$.
- (2) $e^*Cl_{[\gamma, \gamma']}(e^*Cl_{[\gamma, \gamma']}(A)) = e^*Cl_{[\gamma, \gamma']}(A)$.

Proof. (1). By Proposition 3.14, it suffices to prove that $e^*_{[\gamma, \gamma']}Cl(A) \subseteq e^*Cl_{[\gamma, \gamma']}(A)$. Let $x \in e^*_{[\gamma, \gamma']}Cl(A)$ and W and S be e^* -open sets of X containing x . By the e^* -openness of γ and γ' , there exist an e^*_γ -open set W' and an $e^*_{\gamma'}$ -open set S' such that $x \in W' \subseteq W^\gamma$ and $x \in S' \subseteq S^{\gamma'}$. By Propositions 3.6 and 3.12, $(S' \cap W') \cap A \neq \emptyset$ and hence $(S^\gamma \cap W^{\gamma'}) \cap A \neq \emptyset$. This implies that $x \in e^*Cl_{[\gamma, \gamma']}(A)$.

2. This follows immediately from (1) and Proposition 3.13 (3).

Remark 3.7 The below example shows that the equalities of Theorem 3.3 are not true without the assumption that both operations are e^* -open.

Example 3.6 Let $X = \{z_1, z_2, z_3\}$ and $\tau = \{\phi, X, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$. For each $A \in e^*O(X, \tau)$ we define two operations γ and γ' , respectively, by $A^\gamma = Cl(A)$ and $A^{\gamma'} = X$. The operation γ is not e^* -open. However, $e^*Cl_{[\gamma, \gamma']}(\{z_1\}) = \{z_1, z_3\} \subseteq e_{[\gamma, \gamma']}^*Cl(\{z_1\}) = X$ and $e^*Cl_{[\gamma, \gamma]}(e^*Cl_{[\gamma, \gamma']}(\{z_1\})) = X \neq \{z_1, z_3\} = e^*Cl_{[\gamma, \gamma']}(\{z_1\})$.

Theorem 3.4 Let A and B be subsets of a topological space (X, τ) and $\gamma, \gamma': e^*O(X, \tau) \rightarrow P(X)$ operations on $e^*O(X, \tau)$. Then we have the following properties:

- (1) $A \subseteq e^*Cl_{[\gamma, \gamma]}(A)$.
- (2) $e^*Cl_{[\gamma, \gamma]}(\emptyset) = \emptyset$ and $e^*Cl_{[\gamma, \gamma]}(X) = X$.
- (3) $A \in e^*C(X, \tau)_{[\gamma, \gamma]}$ if and only if $e^*Cl_{[\gamma, \gamma]}(A) = A$.
- (4) If $A \subseteq B$, then $e^*Cl_{[\gamma, \gamma]}(A) \subseteq e^*Cl_{[\gamma, \gamma]}(B)$.
- (5) $e^*Cl_{[\gamma, \gamma]}(A \cup B) \subseteq e^*Cl_\gamma(A) \cup e^*Cl_{\gamma'}(B)$.
- (6) If γ and γ' are e^* -regular, then $e^*Cl_{[\gamma, \gamma]}(A \cup B) = e^*Cl_{[\gamma, \gamma]}(A) \cup e^*Cl_{[\gamma, \gamma]}(B)$.
- (7) $e^*Cl_{[\gamma, \gamma]}(A \cap B) \subseteq e^*Cl_{[\gamma, \gamma]}(A) \cap e^*Cl_{[\gamma, \gamma]}(B)$.

Proof. (1), (2) and (4). They are obtained from Definition 3.7.

(3). Suppose that A is $e_{[\gamma, \gamma]}^*$ -closed. Then $X \setminus A$ is $e_{[\gamma, \gamma]}^*$ -open in (X, τ) . We claim that $e^*Cl_{[\gamma, \gamma]}(A) \subseteq A$. Let $x \notin A$. There exist e^* -open sets U and V of (X, τ) containing x such that $U^\gamma \cap V^{\gamma'} \subseteq X \setminus A$, that is, $(U^\gamma \cap V^{\gamma'}) \cap A = \phi$. Hence by Definition 3.7, we have that $x \notin e^*Cl_{[\gamma, \gamma]}(A)$ and so $e^*Cl_{[\gamma, \gamma]}(A) \subseteq A$. By (1), it is proved that $e^*Cl_{[\gamma, \gamma]}(A) = A$. Conversely, suppose that $e^*Cl_{[\gamma, \gamma]}(A) = A$. Let $x \in X \setminus A$. Since $x \notin e^*Cl_{[\gamma, \gamma]}(A)$, there exist an e^* -open sets U and V containing x such that $(U^\gamma \cap V^{\gamma'}) \cap A = \phi$, that is, $U^\gamma \cap V^{\gamma'} \subseteq X \setminus A$. Therefore, A is $e_{[\gamma, \gamma]}^*$ -closed.

(5), (7). They are obtained from (4).

(6). Let $x \notin e^*Cl_{[\gamma, \gamma]}(A) \cup e^*Cl_{[\gamma, \gamma]}(B)$. Then there exist e^* -open sets U, V, W and S of (X, τ) containing x such that $(U^\gamma \cap V^{\gamma'}) \cap A = \phi$ and $(W^\gamma \cap S^{\gamma'}) \cap B = \phi$. Since γ and γ' are e^* -regular, by definition of e^* -regular, there exist e^* -open sets K and L of (X, τ) containing x such that $K^\gamma \subseteq U^\gamma \cap W^\gamma$ and $L^{\gamma'} \subseteq V^{\gamma'} \cap S^{\gamma'}$. Thus, we have $(K^\gamma \cap L^{\gamma'}) \cap (A \cup B) \subseteq ((U^\gamma \cap W^\gamma) \cap (V^{\gamma'} \cap S^{\gamma'})) \cap (A \cup B) = ((U^\gamma \cap V^{\gamma'}) \cap (W^\gamma \cap S^{\gamma'})) \cap (A \cup B) = [((U^\gamma \cap V^{\gamma'}) \cap (W^\gamma \cap S^{\gamma'})) \cap A] \cup [((U^\gamma \cap V^{\gamma'}) \cap (W^\gamma \cap S^{\gamma'})) \cap B] = \phi$, that is, $(K^\gamma \cap L^{\gamma'}) \cap (A \cup B) = \phi$. Hence, $x \notin e^*Cl_{[\gamma, \gamma]}(A \cup B)$. This shows that $e^*Cl_{[\gamma, \gamma]}(A) \cup e^*Cl_{[\gamma, \gamma]}(B) \supseteq e^*Cl_{[\gamma, \gamma]}(A \cup B)$.

Remark 3.8 Example 3.1 shows that the inclusion of Theorem 3.4 (5) is a proper one in general. For a subset $\{w_1\}$, $e^*Cl_{[\gamma, \gamma]}(\{w_1\}) = \{a\} \subseteq e^*Cl_\gamma(\{w_1\}) \cup e^*Cl_{\gamma'}(\{w_1\}) = \{w_1, w_2\}$.

We define the $e_{[\gamma, \gamma]}^*$ -interior of a subset A of (X, τ) as follows:

Definition 3.8 Let A be a subset of a topological space (X, τ) . The union of all $e_{[\gamma, \gamma]}^*$ -open sets contained in A is called the $e_{[\gamma, \gamma]}^*$ -interior of A and is denoted by $e_{[\gamma, \gamma]}^*Int(A)$.

Proposition 3.15 For any subsets A, B of X , we have the following:

- (1) $e_{[\gamma, \gamma]}^*Int(A)$ is an $e_{[\gamma, \gamma]}^*$ -open set in X .
- (2) A is $e_{[\gamma, \gamma]}^*$ -open if and only if $A = e_{[\gamma, \gamma]}^*Int(A)$.
- (3) $e_{[\gamma, \gamma]}^*Int(e_{[\gamma, \gamma]}^*Int(A)) = e_{[\gamma, \gamma]}^*Int(A)$.
- (4) $e_{[\gamma, \gamma]}^*Int(A) \subseteq A$.

- (5) If $A \subseteq B$, then $e^*_{[\gamma, \gamma']} \text{-Int}(A) \subseteq e^*_{[\gamma, \gamma']} \text{-Int}(B)$.
- (6) $e^*_{[\gamma, \gamma']} \text{-Int}(A \cup B) \supseteq e^*_{[\gamma, \gamma']} \text{-Int}(A) \cup e^*_{[\gamma, \gamma']} \text{-Int}(B)$.
- (7) $e^*_{[\gamma, \gamma']} \text{-Int}(A \cap B) \subseteq e^*_{[\gamma, \gamma']} \text{-Int}(A) \cap e^*_{[\gamma, \gamma']} \text{-Int}(B)$.

Proof. Obvious.

Proposition 3.16 Let A be any subset of a topological space (X, τ) . Then the following statements are true:

- (1) $X \setminus e^*_{[\gamma, \gamma']} \text{-Int}(A) = e^*_{[\gamma, \gamma']} \text{-Cl}(X \setminus A)$.
- (2) $X \setminus e^*_{[\gamma, \gamma']} \text{-Cl}(A) = e^*_{[\gamma, \gamma']} \text{-Int}(X \setminus A)$.
- (3) $e^*_{[\gamma, \gamma']} \text{-Int}(A) = X \setminus e^*_{[\gamma, \gamma']} \text{-Cl}(X \setminus A)$.
- (4) $e^*_{[\gamma, \gamma']} \text{-Cl}(A) = X \setminus e^*_{[\gamma, \gamma']} \text{-Int}(X \setminus A)$.

4. $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -Functions

Throughout this section, let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $\gamma, \gamma' : e^*O(X, \tau) \rightarrow P(X)$ be operations on $e^*O(X, \tau)$ and $\beta, \beta' : e^*O(Y, \sigma) \rightarrow P(Y)$ be operations on $e^*O(Y, \sigma)$.

Definition 4.1 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -continuous if for each point $x \in X$ and each e^* -open sets W and S of (Y, σ) containing $f(x)$ there exist e^* -open sets U and V of (X, τ) containing x such that $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$.

Theorem 4.1 Let (1), (2), (3), (4), (5), (6), and (7) be the following properties for a function $f : (X, \tau) \rightarrow (Y, \sigma)$.

- (1) $f : (X, \tau) \rightarrow (Y, \sigma)$ is $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -continuous.
- (2) $f(e^*Cl_{[\gamma, \gamma']}(A)) \subseteq e^*Cl_{[\beta, \beta']}(f(A))$ for every subset A of (X, τ) .
- (3) $e^*Cl_{[\gamma, \gamma']}(f^{-1}(B)) \subseteq f^{-1}(e^*Cl_{[\beta, \beta']}(B))$ for every subset B of (Y, σ) .
- (4) $f^{-1}(B)$ is $e^*_{[\gamma, \gamma']}$ -closed for every $e^*_{[\beta, \beta']}$ -closed set B of (Y, σ) .
- (5) $f(e^*_{[\gamma, \gamma']} \text{-Cl}(A)) \subseteq e^*_{[\beta, \beta']} \text{-Cl}(f(A))$ for every subset A of (X, τ) .
- (6) $f^{-1}(V)$ is $e^*_{[\gamma, \gamma']}$ -open for every $e^*_{[\beta, \beta']}$ -open set V of (Y, σ) .
- (7) For each point $x \in X$ and each $e^*_{[\beta, \beta']}$ -open W of (Y, σ) containing $f(x)$, there exists $e^*_{[\gamma, \gamma']}$ -open U of (X, τ) containing x such that $f(U) \subseteq W$.

Then (1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) hold.

Proof. (1) \Rightarrow (2). Let $x \in e^*Cl_{[\gamma, \gamma']}(A)$ and W, S be e^* -open sets of (Y, σ) containing $f(x)$. There exist e^* -open sets U and V of (X, τ) containing x such that $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. Since $x \in e^*Cl_{[\gamma, \gamma']}(A)$, then $(U^\gamma \cap V^{\gamma'}) \cap A \neq \emptyset$, implies that $f(U^\gamma \cap V^{\gamma'}) \cap f(A) \neq \emptyset$. Therefore, we have $f(A) \cap (W^\beta \cap S^{\beta'}) \neq \emptyset$. Therefore $f(x) \in e^*Cl_{[\beta, \beta']}(f(A))$, which implies that $x \in f^{-1}(e^*Cl_{[\beta, \beta']}(f(A)))$. Hence $e^*Cl_{[\gamma, \gamma']}(A) \subseteq f^{-1}(e^*Cl_{[\beta, \beta']}(f(A)))$, so that $f(e^*Cl_{[\gamma, \gamma']}(A)) \subseteq e^*Cl_{[\beta, \beta']}(f(A))$.

(2) \Rightarrow (3). Let B be any subset of Y . Then $f^{-1}(B)$ is a subset of X . By (2), we have $f(e^*Cl_{[\gamma, \gamma']}(f^{-1}(B))) \subseteq e^*Cl_{[\beta, \beta']}(f(f^{-1}(B))) \subseteq e^*Cl_{[\beta, \beta']}(B)$. Hence $e^*Cl_{[\gamma, \gamma']}(f^{-1}(B)) \subseteq f^{-1}(e^*Cl_{[\beta, \beta']}(B))$.

(3) \Rightarrow (2). Let A be any subset of X . Then $f(A)$ is a subset of Y . By (3), we have $e^*Cl_{[\gamma, \gamma']}(f^{-1}(f(A))) \subseteq f^{-1}(e^*Cl_{[\beta, \beta']}(f(A)))$. This implies that $e^*Cl_{[\gamma, \gamma']}(A) \subseteq f^{-1}(e^*Cl_{[\beta, \beta']}(f(A)))$. Hence $f(e^*Cl_{[\gamma, \gamma']}(A)) \subseteq e^*Cl_{[\beta, \beta']}(f(A))$.

(3) \Rightarrow (4) . Let B be an $e_{[\beta, \beta']}^*$ -closed set of (Y, σ) . By (3) and Theorem 3.1, $e^*Cl_{[\gamma, \gamma']}(f^{-1}(B)) \subseteq f^{-1}(B)$ and hence $f^{-1}(B)$ is $e_{[\gamma, \gamma']}^*$ -closed.

(4) \Rightarrow (5) . Let A be any subset of X . Then $f(A) \subseteq e_{[\beta, \beta']}^*Cl(f(A))$ and $e_{[\beta, \beta']}^*Cl(f(A))$ is an $e_{[\beta, \beta']}^*$ -closed set in Y . Hence $A \subseteq f^{-1}(e_{[\beta, \beta']}^*Cl(f(A)))$. By (4), we have $f^{-1}(e_{[\beta, \beta']}^*Cl(f(A)))$ which is an $e_{[\gamma, \gamma']}^*$ -closed set in X . Therefore, $e_{[\gamma, \gamma']}^*Cl(A) \subseteq f^{-1}(e_{[\beta, \beta']}^*Cl(f(A)))$. Hence $f(e_{[\gamma, \gamma']}^*Cl(A)) \subseteq e_{[\beta, \beta']}^*Cl(f(A))$.

(5) \Rightarrow (4) . Let B be an $e_{[\beta, \beta']}^*$ -closed set of (Y, σ) . By (5), $e_{[\gamma, \gamma']}^*Cl(f^{-1}(B)) \subseteq f^{-1}(e_{[\beta, \beta']}^*Cl(f(f^{-1}(B)))) \subseteq f^{-1}(e_{[\beta, \beta']}^*Cl(B)) \subseteq f^{-1}(B)$. Therefore, by Proposition 3.13, $f^{-1}(B)$ is $e_{[\gamma, \gamma']}^*$ -closed.

(5) \Leftrightarrow (6) . This follows from Definition 3.5 and the equivalence of (4) \Leftrightarrow (5) .

(6) \Rightarrow (7) . Let W be any $e_{[\beta, \beta']}^*$ -open set in Y containing $f(x)$, so its inverse image is an $e_{[\gamma, \gamma']}^*$ -open set in X . Since $f(x) \in W$, then $x \in f^{-1}(W)$ and by hypothesis $f^{-1}(W)$ is an $e_{[\gamma, \gamma']}^*$ -open set in X containing x so that $f(f^{-1}(W)) \subseteq W$.

(7) \Rightarrow (6) . Let $V \in e^*O(Y, \sigma)_{[\beta, \beta']}$. For each $x \in f^{-1}(V)$, by (7), there exists an $e_{[\gamma, \gamma']}^*$ -open set U_x containing x such that $f(U_x) \subseteq V$. Then we have $f^{-1}(V) = \bigcup \{U_x \in e^*O(X, \tau)_{[\gamma, \gamma']} : x \in f^{-1}(V)\}$ and hence $f^{-1}(V) \in e^*O(X, \tau)_{[\gamma, \gamma']}$ using Proposition 3.1 \square

Corollary 4.1 If (Y, σ) is an $e_{[\beta, \beta']}^*$ -regular space, or operations β and β' are e^* -open on $e^*O(Y, \sigma)$, then all properties of Theorem 4.1 are equivalent.

Proof. By Theorem 4.1, it is sufficient to prove the implication (4) \Rightarrow (1) , where (1) and (4) are the properties of Theorem 4.1

First, we show the implication under the assumption that (Y, σ) is an $e_{[\beta, \beta']}^*$ -regular space. Let $x \in X$ and W, S be e^* -open sets of (Y, σ) containing $f(x)$. By Proposition 3.10, $Y \setminus (W \cap S)$ is $e_{[\beta, \beta']}^*$ -closed. Then, $f^{-1}(Y \setminus (W \cap S))$ is $e_{[\gamma, \gamma']}^*$ -closed by (4) and hence $f^{-1}(W \cap S)$ is $e_{[\gamma, \gamma']}^*$ -open set of (X, τ) containing x . Therefore, there exist e^* -open sets U and V of (X, τ) containing x such that $U^\gamma \cap V^{\gamma'} \subseteq f^{-1}(W \cap S)$ and so $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. This implies that f is $(e_{[\gamma, \gamma']}^*, e_{[\beta, \beta']}^*)$ -continuous.

Second, we suppose that the operations β and β' are e^* -open. Let $x \in X$ and W, S be e^* -open sets of (Y, σ) containing $f(x)$. By using e^* -openness of β and β' , there exist an e_β^* -open set A and an $e_{\beta'}^*$ -open set B such that $f(x) \in A \cap B$ and $A \cap B \subseteq W^\beta \cap S^{\beta'}$. By Proposition 3.6, $Y \setminus (A \cap B)$ is $e_{[\gamma, \gamma']}^*$ -closed and hence $f^{-1}(Y \setminus (A \cap B))$ is $e_{[\beta, \beta']}^*$ -closed. Therefore, there exist e^* -open sets U and V of (X, τ) containing x such that $U^\gamma \cap V^{\gamma'} \subseteq f^{-1}(A \cap B)$ and so $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. This implies that f is $(e_{[\gamma, \gamma']}^*, e_{[\beta, \beta']}^*)$ -continuous. \square

Remark 4.1 It is clear that the identity operation and the operation $X : e^*O(X, \tau) \rightarrow (X, \tau)$ are e^* -open on $e^*O(X, \tau)$. Therefore, by Corollary 4.1, if β and β' are chosen from mentioned operations above, then all properties of Theorem 4.1 are equivalent.

Remark 4.2 The converse of implication (1) \Rightarrow (4) in Theorem 4.1 is not true in general as shown by the following example.

Example 4.1 Let $X = \{s_1, s_2, s_3\}$ and $\tau = \{\phi, X, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$ be a topology on X . Let $f : (X, \tau) \rightarrow (X, \tau)$ be the identity. Let $\gamma = \gamma' = \beta' = X : e^*O(X, \tau) \rightarrow P(X)$ be the operations on $e^*O(X, \tau)$ and β the closure operation on $e^*O(X, \tau)$. Then, the condition (4) in Theorem 4.1 is true. It is shown that f is not $(e_{[\gamma, \gamma']}^*, e_{[\beta, \beta']}^*)$ -continuous.

Theorem 4.1 suggests the following.

Remark 4.3 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $(e_{[\gamma, \gamma']}^*, e_{[\beta, \beta']}^*)$ -continuous, then the induced function $f : (X, e^*O(X, \tau)_{[\gamma, \gamma']}) \rightarrow (Y, e^*O(Y, \sigma)_{[\beta, \beta]})$ is continuous.

Remark 4.4 The converse of Remark 4.3 is not true in general as shown by the following example.

Example 4.2 Let (X, τ) as in Example 4.1 and $f : (X, \tau) \rightarrow (X, \tau)$ be a function defined by

$$f(x) = \begin{cases} s_2 & \text{if } x = s_1, \\ s_3 & \text{if } x = s_2, \\ s_1 & \text{if } x = s_3, \end{cases}$$

moreover let $\gamma = \beta$ be the closure operation on $e^*O(X, \tau)$ and $\gamma' = \beta' = X : e^*O(X, \tau) \rightarrow P(X)$. Then, $e^*O(X, \tau)_{[\gamma, X]} = \{\emptyset, X\}$ and it is shown that f is not $(e^*_{[\gamma, X]}, e^*_{[\gamma, X]})$ -continuous. However, $f : (X, e^*O(X, \tau)_{[\gamma, \gamma']}) \rightarrow (Y, e^*O(Y, \sigma)_{[\beta, \beta']})$ is continuous.

Let $(X, \tau), (Y, \sigma)$ and (Z, η) be spaces and $\gamma, \gamma' : e^*O(X, \tau) \rightarrow P(X)$, $\beta, \beta' : e^*O(Y) \rightarrow P(Y)$ and $\delta, \delta' : e^*O(Z) \rightarrow P(Z)$, be operations on $e^*O(X, \tau)$, $e^*O(Y, \sigma)$ and $e^*O(Z, \eta)$, respectively.

Theorem 4.2 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $(e^*_{[\beta, \beta']}, e^*_{[\delta, \delta']})$ -continuous, then its composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is $(e^*_{[\gamma, \gamma']}, e^*_{[\delta, \delta']})$ -continuous.

Proof. Let $x \in X, K$ and L be e^* -open sets of Z containing $g(f(x))$. Since g is $(e^*_{[\beta, \beta']}, e^*_{[\delta, \delta']})$ -continuous, then there exist e^* -open sets W and S of Y containing $f(x)$ such that $g(W^\beta \cap S^{\beta'}) \subseteq K^\delta \cap L^{\delta'}$. Also, since f is $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -continuous, then there exist e^* -open sets U and V of X containing x such that $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'}$. This implies that $f(U^\gamma \cap V^{\gamma'}) \subseteq W^\beta \cap S^{\beta'} \subseteq g^{-1}(K^\delta \cap L^{\delta'})$. Then we obtain $(g \circ f)(U^\gamma \cap V^{\gamma'}) \subseteq K^\delta \cap L^{\delta'}$. Therefore, $g \circ f$ is $(e^*_{[\gamma, \gamma']}, e^*_{[\delta, \delta']})$ -continuous. \square

Definition 4.2 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -closed if for $e^*_{[\gamma, \gamma']}$ -closed set A of X , $f(A)$ is $e^*_{[\beta, \beta']}$ -closed in Y .

Proposition 4.1 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -closed function. Then, for each subset B of (Y, σ) and each $e^*_{[\gamma, \gamma']}$ -open set U containing $f^{-1}(B)$, there exists an $e^*_{[\beta, \beta']}$ -open set V such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Let $V = Y \setminus f(X \setminus U)$. Then V is $e^*_{[\beta, \beta']}$ -open. Thus $f^{-1}(B) \subseteq U$ implies $B \subseteq V$ and $f^{-1}(V) = f^{-1}(Y \setminus f(X \setminus U)) = X \setminus f^{-1}(f(X \setminus U)) \subseteq X \setminus (X \setminus U) = U$, or $f^{-1}(V) \subseteq U$. \square

Proposition 4.2 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is bijective and $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is $(e^*_{[\beta, \beta']}, e^*_{[\gamma, \gamma']})$ -continuous, then f is $(e^*_{[\gamma, \gamma']}, e^*_{[\beta, \beta']})$ -closed.

Proof. This follows from definitions and Theorem 4.1

5. Application to Share Market: An Illustrative Example

In this section, we adapt the concept of $e^*_{[\gamma, \gamma']}$ -open sets to a share market context, providing a practical illustration inspired by Example 3.1. This example demonstrates how trader reactions to stock price ranges can determine their stability for trading purposes.

Example: Stable Price Ranges in a Share Market. Consider a share market with three stocks, $X = \{s_1, s_2, s_3\}$, representing Stock 1 (e.g., a technology company), Stock 2 (e.g., an energy company), and Stock 3 (e.g., a retail company). The topology τ defines observable bullish price ranges:

$$\tau = \{\emptyset, X, \{s_2\}, \{s_3\}, \{s_1, s_2\}, \{s_2, s_3\}\},$$

where \emptyset indicates no stocks in focus, X is the entire market, $\{s_2\}$ and $\{s_3\}$ denote individual stocks trending upward, and $\{s_1, s_2\}$ and $\{s_2, s_3\}$ represent pairs of stocks in a bullish trend.

We interpret e^* -open sets as "reliable price ranges" where stock price movements are consistent and predictable, assuming for simplicity that $e^*O(X, \tau)$ includes all sets in τ . Two operations model trader reactions:

$$\gamma(A) = \begin{cases} A & \text{if } s_3 \in A, \\ X & \text{if } s_3 \notin A, \end{cases} \quad \text{and} \quad \gamma'(A) = \begin{cases} A & \text{if } A \neq \{s_2\}, \\ X & \text{if } A = \{s_2\}. \end{cases}$$

Here, γ represents the reaction of conservative traders: if Stock 3 (retail) is trending up in A , they focus only on A ; if absent, they assume a market-wide impact (X). Operation γ' models speculative traders: they stick to A unless it's just Stock 2 (energy), triggering a market-wide reaction (X) due to its perceived volatility.

A price range A is $e^*_{[\gamma, \gamma']}$ -open (Definition 3.1) if, for each stock $x \in A$, there exist e^* -open sets U and V containing x , where U is the conservative traders' observed range and V is the speculative traders' observed range, such that $U^\gamma \cap V^{\gamma'} \subseteq A$. This indicates a "stable trading range" where combined trader reactions do not exceed the intended focus.

- Consider $A = \{s_1, s_2\}$ (tech and energy trending up):
 - For $x = s_1$: Conservative traders observe $U = \{s_1, s_2\}$ (no s_3 , so $U^\gamma = X$), and speculative traders observe $V = \{s_1, s_2\}$ ($V^{\gamma'} = \{s_1, s_2\}$ as $V \neq \{s_2\}$). Then $U^\gamma \cap V^{\gamma'} = X \cap \{s_1, s_2\} = \{s_1, s_2\} \subseteq \{s_1, s_2\}$.
 - For $x = s_2$: Using the same U and V , $U^\gamma \cap V^{\gamma'} = \{s_1, s_2\} \subseteq \{s_1, s_2\}$.

Thus, $\{s_1, s_2\}$ is $e^*_{[\gamma, \gamma']}$ -open, a stable range where trader reactions align with the tech-energy pair.

- Consider $A = \{s_2, s_3\}$ (energy and retail trending up):
 - For $x = s_2$: Conservative traders observe $U = \{s_2, s_3\}$ ($U^\gamma = \{s_2, s_3\}$ as $s_3 \in U$), speculative traders observe $V = \{s_2, s_3\}$ ($V^{\gamma'} = \{s_2, s_3\}$). Then $U^\gamma \cap V^{\gamma'} = \{s_2, s_3\} \subseteq \{s_2, s_3\}$.
 - For $x = s_3$: The same holds.

Thus, $\{s_2, s_3\}$ is $e^*_{[\gamma, \gamma']}$ -open, stable for energy-retail trading.

- Consider $A = \{s_2\}$ (energy alone), the intersection of $\{s_1, s_2\}$ and $\{s_2, s_3\}$:
 - For $x = s_2$: Conservative traders observe $U = \{s_2\}$ ($U^\gamma = X$ as $s_3 \notin U$), speculative traders observe $V = \{s_2\}$ ($V^{\gamma'} = X$ as $V = \{s_2\}$). Then $U^\gamma \cap V^{\gamma'} = X \cap X = X \not\subseteq \{s_2\}$. Alternatively, if $U = \{s_1, s_2\}$ ($U^\gamma = X$) and $V = \{s_2\}$ ($V^{\gamma'} = X$), still $X \not\subseteq \{s_2\}$.

No $U, V \in \tau$ containing s_2 keep $U^\gamma \cap V^{\gamma'} \subseteq \{s_2\}$, so $\{s_2\}$ is not $e^*_{[\gamma, \gamma']}$ -open.

This parallels Example 3.1, where $\{w_1, w_2\}$ and $\{w_2, w_3\}$ are $e^*_{[\gamma, \gamma']}$ -open, but $\{w_2\}$ is not. In this market, $\{s_1, s_2\}$ and $\{s_2, s_3\}$ are stable trading ranges, as conservative and speculative reactions remain contained. However, $\{s_2\}$ alone is unstable—traders overreact, expanding focus to X , disrupting targeted trading on Stock 2. This example highlights how $e^*_{[\gamma, \gamma']}$ -open sets can identify stable financial patterns, bridging topological theory with market dynamics.

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