Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 18** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.82895

Domination and Minimum Energy Problem in Linear Time-Varying Systems

Amissi Chadi, Magri El Mostafa and Lhous Mustapha

Abstract: This paper investigates the notion of domination in time-varying linear perturbed systems. The primary objective of this work is to study the comparison (or classification) of input operators, with respecting the output one. We present the characterization and property results of this concept. We study the optimal control which ensures the domination of time-varying disturbed systems.

Keywords: Time-varying systems, domination, observation, optimal control, input operator.

Contents

1 Introduction **1**

2 Statement of the problem **1**

3 Characterization results **3**

4 Optimal control **7**

5 Minimum energy problem **9**

6 Conclusion **11**

1. Introduction

In virtually all work in systems analysis, the problems considered consist of studying whether a system is controllable, observable, etc., or not, but without studying the possibility of comparing (classifying) the input or output operators themselves. It is for this purpose that the notion of domination was introduced and studied (see [2], [10] and [13]).

The notion of domination was introduced and studied separately for controlled and observed distributed systems. In the field of control theory, it is undeniable that some controls outperform others. This perspective opens up a vast field for the classification of input operators, ultimately giving rise to the following idea: a concept known as domination.

Domination was introduced and developed for a class of systems in the parabolic and hyperbolic cases (see [1]). The aim is to explore the potential for categorizing input operators and output operators based on duality. An extension of domination to the regional case is discussed. The regional aspect of this problem stems from the fact that one system can dominate another in a region, but not on the basis of the system's overall geometry.

The structure of this paper is as follows: In Section 2, we present the model of disturbed time-varying systems, we define the problem statement and we show some conditions to characterize the domination system. In Section 3, the minimum energy problem are described. Finally, section 4 summarizes the conclusion.

2. Statement of the problem

Let's consider the time-variable linear systems expressed by

$$\begin{cases} \dot{z}(\rho) = A(\rho)z(\rho) + B_1(\rho)u_1(\rho) + B_2(\rho)u_2(\rho)\ ;\ 0 < \rho < \Omega \\ z(0) = z_0 \end{cases} \tag{2.1}$$

2020 *Mathematics Subject Classification:* 93C05, 93C41.

Submitted March 21, 2026. Published June 19, 2026.

where $A \in C^{\infty}([0,\Theta], M_n(\mathbb{R}))$, $B_1 \in C^{\infty}([0,\Theta], M_{n,p}(\mathbb{R}))$, $B_2 \in C^{\infty}([0,\Theta], M_{n,m}(\mathbb{R}))$, $u_1 \in L^2(0,\Omega;\mathbb{R}^p)$ and $u_2 \in L^2(0,\Omega;\mathbb{R}^m)$.

The corresponding output is given by

$$y(\rho) = Y(\rho)z(\rho)\ ; \quad 0 < \rho < T \tag{2.2}$$

with $Y \in C^{\infty}([0,\Theta], M_{q,n}(\mathbb{R}))$, we have

$$z(\rho) = \Phi(\rho,0)z_0 + H_1(\rho)u_1 + H_2(\rho)u_2$$

where $\Phi(\rho,0)$ is that nonsingular matrix solution of the homogeneous part of (2.1) such that $\Phi(0,0) = Id_n$ (Id_n denotes the identity map of $\mathbb{R}^n$).

Then

$$y(\rho) = Y(\rho)\Phi(\rho,0)z_0 + Y(\rho)H_1(\rho)u_1 + Y(\rho)H_2(\rho)u_2$$

where $H_1(\rho)$ and $H_2(\rho)$ are the operators defined by

$$\begin{array}{rccl} H_1(\rho): & L^2(0,\rho;\mathbb{R}^p) & \longrightarrow & \mathbb{R}^n \\ & u_1 & \longrightarrow & \displaystyle\int_0^{\rho} \Phi(\rho,s)B_1(s)u_1(s)ds \end{array} \tag{2.3}$$

and

$$\begin{array}{rccl} H_2(\rho): & L^2(0,\rho;\mathbb{R}^m) & \longrightarrow & \mathbb{R}^n \\ & u_2 & \longrightarrow & \displaystyle\int_0^{\rho} \Phi(\rho,s)B_2(s)u_2(s)ds \end{array} \tag{2.4}$$

For $i = 1, 2$, we note $H_i = H_i(\Omega)$. Then

$$Y(\Omega) = Y(\Omega)\Phi(\Omega,0)z_0 + Y(\Omega)H_1u_1 + Y(\Omega)H_2u_2.$$

Let us define that

$\mathcal{R}(.)$ indicates the range of a operator.

$\mathcal{N}(.)$ indicates the null space of a operator.

We define hereafter the notion of $Y(\Omega)$-domination.

Definition 2.1
We denote that $B_1(\rho)$ dominates $B_2(\rho)$ with respecting to $Y(\Omega)$ on $[0,\Theta]$, if for any $u_2 \in L^2(0,\Omega;\mathbb{R}^m)$, there exists a control $u_1 \in L^2(0,\Omega;\mathbb{R}^p)$ such that:

$$Y(\Omega)H_1u_1 + Y(\Omega)H_2u_2 = 0.$$

In this case, for Ω fixed, one can note $B_2(\rho) \underset{Y(\Omega)}{\leq} B_1(\rho)$.

3. Characterization results

Proposition 3.1 *The subsequent attributes are analogous*

i) B_1 *dominates* B_2 *on* $[0,\Theta]$ *with respect to* $Y(\Omega)$.

ii) $\mathcal{R}(Y(\Omega)H_2) \subset \mathcal{R}(Y(\Omega)H_1)$

iii) $\mathcal{N}(H_1^*Y(\Omega)^*) \subset \mathcal{N}(H_2^*Y(\Omega)^*)$

iv) $\exists \gamma > 0,\ \forall \theta \in \mathbb{R}^q$,

$$\| B_2(.)^*\Phi(\Omega,.)^*Y(\Omega)^*\theta\|_{L^2(0,\Omega;\mathbb{R}^m)} \leq \gamma \, \|B_1(.)^*\Phi(\Omega,.)^*Y(\Omega)^*\theta\|_{L^2(0,\Omega;\mathbb{R}^p)} \tag{3.1}$$

Proof: Infer from the definition that

$$\mathcal{N}(H_i^*Y(\Omega)^*) = \mathcal{N}(B_i(.)^*\Phi(\Omega,.)^*Y(\Omega)^*), \quad for \quad i = 1,2$$

and also the theorem 3.3, page 60 [5].

□

Let us define, by induction on j a sequence of maps $(B_j(\rho))_{0\leq j\leq n-1}$ in the following way:

$$B_{i,0}(\rho) = B_i(\rho), \qquad B_{i,j}(\rho) = -A(\rho)B_{i,j-1}(\rho) + \dot{B}_{i,j-1}(\rho), \qquad \forall\, i = 1,2 \qquad \forall\, j = 1,...,n-1, \tag{3.2}$$

Proposition 3.2
Assume that, for some $\bar{\rho} \in [0,\Theta]$,

$$rank \begin{pmatrix} Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,0}(\bar{\rho}) & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,1}(\bar{\rho}) & ... & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,n-1}(\bar{\rho}) \end{pmatrix} = q,$$

then B_1 *dominates* B_2 *on* $[0,\Theta]$ *with respect to* $Y(\Omega)$.

The proof of proposition 3.2 is based on the following result.

Lemma 3.1 *Assume that, for some* $\bar{\rho} \in [0,\Theta]$,

$$rank \begin{pmatrix} Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,0}(\bar{\rho}) & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,1}(\bar{\rho}) & ... & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,n-1}(\bar{\rho}) \end{pmatrix} = q,$$

then the operator defined by

$$\begin{array}{rccl} Y(\Omega)H_1 : & L^2(0,\Omega;\mathbb{R}^p) & \longrightarrow & \mathbb{R}^q \\ & u_1 & \longrightarrow & \displaystyle\int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)u_1(s)ds \end{array}$$

is surjective.

Proof: We assume that the operator $Y(\Omega)H_1$ is not surjective, then there exists a certain row vector $\Psi \in \mathbb{R}^q\backslash\{0\}$ such that for any $u_1 \in L^2(0,\Omega;\mathbb{R}^p)$, one gets

$$\Psi \int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)u_1(s)ds = 0,$$

from which we get

$$\Psi Y(\Omega)\Phi(\Omega,s)B_1(s) = 0,$$

one gets

$$K(s) = \Psi Y(\Omega)\Phi(\Omega,s)B_1(s) = 0, \qquad \forall\, s \in [0,\Theta]. \tag{3.3}$$

Hence Ψ, is not 0, differentiating (3.3) i times, where $0 \le i \le n-1$, using (3.2) and an induction argument on i, one gets

$$K^{(i)}(s) = \Psi Y(\Omega)\Phi(\Omega,s)B_{1,i}(s) = 0, \qquad \forall\, s \in [0,\Theta], \qquad \forall\, i \in \{0,...,n-1\}, \tag{3.4}$$

hence (3.3) and (3.4) implies that

$$\Psi Y(\Omega)\Phi(\Omega,s)B_{1,i}(s) = 0, \qquad \forall\, s \in [0,\Theta], \qquad \forall\, i \in \{0,...,n-1\},$$

this concludes our proof of lemma 3.1. □

Proof: We assume that, for some $\bar{\rho} \in [0,\Theta]$,

$$rank\left(\begin{array}{cccc} Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,0}(\bar{\rho}) & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,1}(\bar{\rho}) & \dots & Y(\Omega)\Phi(\Omega,\bar{\rho})B_{1,n-1}(\bar{\rho})\end{array}\right) = q,$$

then, by lemma 3.1, the operator $Y(\Omega)H_1$ is surjective, i.e, $\mathcal{R}(Y(\Omega)H_1) = \mathbb{R}^q$, that implies

$$\mathcal{R}(Y(\Omega)H_2) \subset \mathcal{R}(Y(\Omega)H_1),$$

then B_1 dominates B_2 on $[0,\Theta]$ with respect to $Y(\Omega)$. □

The next theorem gives a characterization result.

Theorem 3.1
Suppose $A(\rho), B_1(\rho), B_2(\rho)$ and $Y(\rho)$ are analytic, B_1 dominates B_2 on $[0,\Theta]$ with respect to $Y(\Omega)$ if and only if for every $\rho \in [0,\Theta]$

$$\mathcal{R}\left(\begin{array}{cccc} Y(\Omega)\Phi(\Omega,\rho)B_{2,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{2,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{2,n-1}(\rho)\end{array}\right) \subset$$

$$\mathcal{R}\left(\begin{array}{cccc} Y(\Omega)\Phi(\Omega,\rho)B_{1,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{1,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{1,n-1}(\rho)\end{array}\right).$$

The proof of theorem 3.1 is based on the following results.

Lemma 3.2 $\mathcal{R}(\bar{\Theta}_i(\Omega)) = \mathcal{R}(Y(\Omega)H_i)$, *for $i = 1,2$, with*

$$\bar{\Theta}_i(\Omega) = \int_0^{\Omega} Y(\Omega)\Phi(\Omega,s)B_i(s)B_i(s)^*\Phi(\Omega,s)^*Y(\Omega)^*ds.$$

Proof: Let $i = 1,2$. We first show that $\mathcal{R}(\bar{\Theta}_i(\Omega)) \subset \mathcal{R}(Y(\Omega)H_i)$. Let $x_i \in \mathcal{R}(\bar{\Theta}_i(\Omega))$; that is, there exists $\Psi \in \mathbb{R}^q$ such that $\bar{\Theta}_i(\Omega)\Psi = x_i$. Choose $u_i(s) = H_i^*Y(\Omega)^*\Psi = B_i^*(s)\Phi(\Omega,s)^*Y(\Omega)^*\Psi$. Then

$$Y(\Omega)H_iu_i = \int_0^{\Omega} Y(\Omega)\Phi(\Omega,s)B_i(s)B_i^*(s)\Phi(\Omega,s)^*Y(\Omega)^*\Psi ds = \bar{\Theta}_i(\Omega)\Psi = x_i.$$

Therefore, $x_i \in \mathcal{R}(Y(\Omega)H_i)$, and since x_i is arbitrary, it follows that $\mathcal{R}(\bar{\Theta}_i(\Omega)) \subset \mathcal{R}(Y(\Omega)H_i)$.

We shall now show that $\mathcal{R}(Y(\Omega)H_i) \subset \mathcal{R}(\bar{\Theta}_i(\Omega))$, which together with $\mathcal{R}(\bar{\Theta}_i(\Omega)) \subset \mathcal{R}(Y(\Omega)H_i)$ proves that $\mathcal{R}(\bar{\Theta}_i(\Omega)) = \mathcal{R}(Y(\Omega)H_i)$. Let $x_i \in \mathcal{R}(Y(\Omega)H_i)$, i.e., there exists u_i such that $Y(\Omega)H_iu_i = x_i$. We assume that $x_i \notin \mathcal{R}(\bar{\Theta}_i(\Omega))$ and we shall show that this leads to a contradiction. This implies that the null space of $\bar{\Theta}_i(\Omega)$ is nonempty. $\bar{\Theta}_i(\Omega)$ is symmetric, and so the range of $\bar{\Theta}_i(\Omega)$ is the orthogonal complement of its null space.
Thus for any $\alpha_i \in \mathcal{R}(\bar{\Theta}_i(\Omega))$ and $\beta_i \in \mathcal{N}(\bar{\Theta}_i(\Omega))$, $\alpha_i^*\beta_i = 0$. Also, we may write $x_i = x_{i,1} + x_{i,2}$ with

$x_{i,1} \in \mathcal{R}(\bar{\Theta}_i(\Omega))$ and $x_{i,2} \in \mathcal{N}(\bar{\Theta}_i(\Omega))$ ($x_{i,2} \neq 0$ since $x_i \notin \mathcal{R}(\bar{\Theta}_i(\Omega))$). Then there exists $\delta_i \in \mathcal{N}(\bar{\Theta}_i(\Omega))$ such that $\delta_i^* x_{i,2} \neq 0$, which implies $\delta_i^* x_i \neq 0$. Now, $\delta_i^* \bar{\Theta}_i(\Omega)\delta_i = 0$, that is,

$$\int_0^{\Omega} \delta_i^* Y(\Omega)\Phi(\Omega,s)B_i(s)B_i(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\delta_i ds = 0. \tag{3.5}$$

Then (3.5) is equal to

$$\int_0^{\Omega} \|B_i(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\delta_i\|^2 ds.$$

Then

$$\delta_i^* Y(\Omega)\Phi(\Omega,s)B_i(s) = 0, \qquad s \in [0,\Theta].$$

This in turn implies that

$$\delta_i^* x_i = \delta_i^* Y(\Omega)H_i u_i = \int_0^{\Omega} \delta_i^* Y(\Omega)\Phi(\Omega,s)B_i(s)u_i(s)ds = 0,$$

which is a contradiction since $\delta_i^* x_i \neq 0$. Therefore, $x_i \in \mathcal{R}(\bar{\Theta}_i(\Omega))$, which implies that $\mathcal{R}(Y(\Omega)H_i) \subset \mathcal{R}(\bar{\Theta}_i(\Omega))$.

□

Lemma 3.3
Suppose $A(\rho), B_1(\rho), B_2(\rho)$ and $Y(\rho)$ are analytic, we have for $i = 1,2$

$$\mathcal{R}(\bar{\Theta}_i(\Omega)) = \mathcal{R}\left(\begin{array}{cccc} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho) \end{array}\right),$$

for every $\rho \in [0,\Theta]$.

The proof of lemma 3.3 a consequence of the following lemma [4].

Lemma 3.4 *Suppose $A(\rho), B_1(\rho)$ and $B_2(\rho)$ are analytic, let for $i = 1,2$*

$$Q_j(\rho) = [B_{i,0}(\rho), B_{i,1}(\rho), ..., B_{i,j}(\rho)], \qquad j = 0,1,...$$

Then there exists $k \leq n-1$, and non-empty open set $O \in [0,M]$, where $M > 0$, such that for each $\rho \in O$,

$$rank\left(Q_k(\rho)\right) = rank\left(Q_{k+j}(\rho)\right), \qquad j = 1,2,...$$

Proof: Let $i = 1,2$. We first show that for every ρ

$$\mathcal{R}(\bar{\Theta}_i(\Omega)) \subset Im\left(\begin{array}{cccc} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho) \end{array}\right).$$

Let

$$\Psi_{i,\rho} \in \mathcal{N}\begin{pmatrix} B_{i,0}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \\ B_{i,1}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \\ \vdots \\ B_{i,n-1}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \end{pmatrix},$$

Lemma 3.4 implies that the columns of $B_{i,j}(\rho)$ for all $j > n-1$ are, for every $\rho \in O$, expressible as linear combinations of the columns of $[B_{i,0}(\rho), B_{i,1}(\rho), ..., B_{i,n-1}(\rho)]$.
Let O be the set in Lemma 3.4 and choose $\rho_1 \in O$. We have

$$\frac{d^j\left(\Psi_i^* Y(\Omega)\Phi(\Omega,\rho)B_i(\rho)\right)}{d\rho^j}(\rho_1) = \Psi_i^* Y(\Omega)\Phi(\Omega,\rho_1)B_{i,j}(\rho_1) = 0, \qquad j = 0,1,...$$

Since, $A(\rho), B_1(\rho), B_2(\rho)$ and $Y(\rho)$ are analytic, then $\Psi_i^* Y(\Omega)\Phi(\Omega,\rho)B_i(\rho)$ are analytic for $i = 1,2$. One gets

$$B_i(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^*\Psi_i=0, \qquad \forall\rho\in[0,\Theta],$$

which implies that

$$\int_0^\Omega Y(\Omega)\Phi(\Omega,\rho)B_i(\rho)B_i(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^*\Psi_i d\rho=0,$$

from which we get $\Psi_i\in\mathcal{N}(\bar\Theta_i(\Omega))$, it follows that

$$\mathcal{R}(\bar\Theta_i(\Omega))\subset\mathcal{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}.$$

We shall now show that

$$\mathcal{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}\subset\mathcal{R}(\bar\Theta_i(\Omega)).$$

Let $x_i\in\mathbb{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}$, i.e., there exists $\lambda\in\mathbb{R}^{np}$ such that

$$\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}\lambda=x_i.$$

We assume that $x_i\notin\mathcal{R}(\bar\Theta_i(\Omega))$ for some $T>0$, and we shall show that this leads to a contradiction. This implies that the null space of $\bar\Theta_i(\Omega)$ is nonempty. $\bar\Theta_i(\Omega)$ is symmetric, and so the range of $\bar\Theta_i(\Omega)$ is the orthogonal complement of its null space.
Thus for any $\alpha_i\in\mathcal{R}(\bar\Theta_i(\Omega))$ and $\beta_i\in\mathcal{N}(\bar\Theta_i(\Omega))$, $\alpha_i^*\beta_i=0$. Also, we may write $x_i=x_{i,1}+x_{2,0}$ with $x_{i,1}\in\mathcal{R}(\bar\Theta_i(\Omega))$ and $x_{i,2}\in\mathcal{N}(\bar\Theta_i(\Omega))$ ($x_{i,2}\neq0$ since $x_i\notin\mathcal{R}(\bar\Theta_i(\Omega))$). Then there exists $\delta_i\in\mathcal{N}(\bar\Theta_i(\Omega))$ such that $\delta_i^*x_{i,2}\neq0$, which implies $\delta_i^*x_i\neq0$. Now, $\delta_i^*\bar\Theta_i(\Omega)\delta_i=0$, that is,

$$\int_0^\Omega \delta_i^*Y(\Omega)\Phi(\Omega,s)B_i(s)B_i(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\delta_i ds=0. \tag{3.6}$$

Then (3.6) is equal to

$$\int_0^\Omega \|B_i(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\delta_i\|^2ds.$$

Hence (3.6) implies that

$$\delta_i^*Y(\Omega)\Phi(\Omega,s)B_i(s)=0, \qquad \forall s\in[0,\Theta].$$

one gets

$$K(s)=\delta_i^*Y(\Omega)\Phi(\Omega,s)B_i(s)=0, \qquad \forall\, s\in[0,\Theta]. \tag{3.7}$$

Differentiating (3.7) l times, where $0\le l\le n-1$, using (3.2) and an induction argument on l, one gets

$$K^{(l)}(s)=\delta_i^*Y(\Omega)\Phi(\Omega,s)B_{i,l}(s)=0, \qquad \forall\, s\in[0,\Theta], \qquad \forall\, l\in\{0,\dots,n-1\}, \tag{3.8}$$

hence (3.7) and (3.8) implies that

$$\delta_i^*Y(\Omega)\Phi(\Omega,s)B_{i,l}(s)=0, \qquad \forall\, s\in[0,\Theta], \qquad \forall\, l\in\{0,\dots,n-1\}.$$

This in turn implies that

$$\delta_i^*x_i=\delta_i^*\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}\lambda=0,$$

which is a contradiction since $\delta_i^*x_i\neq0$. Therefore, $x_i\in\mathcal{R}(\bar\Theta_i(\Omega))$, which implies that

$$\mathbb{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{i,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{i,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{i,n-1}(\rho)\end{pmatrix}\subset\mathcal{R}(\bar\Theta_i(\Omega)).$$

□

Proof: Let us now give a proof of theorem 3.1. Using the characterization proposition, we have B_1 dominates B_2 on $[0,\Theta]$ with respect to $Y(\Omega)$ if and only if

$$\mathcal{N}(H_1^* Y(\Omega)^*) \subset \mathcal{N}(H_2^* Y(\Omega)^*).$$

Using lemma 3.2 and lemma 3.3, we deduce for $i=1,2$

$$\mathcal{N}\begin{pmatrix} B_{i,0}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \\ B_{i,1}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \\ \vdots \\ B_{i,n-1}(\rho)^*\Phi(\Omega,\rho)^*Y(\Omega)^* \end{pmatrix} = \mathcal{N}(H_i^* Y(\Omega)^*).$$

Consequently, B_1 dominates B_2 on $[0,\Theta]$ with respect to $Y(\Omega)$ if and only if

$$\mathcal{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{2,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{2,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{2,n-1}(\rho) \end{pmatrix} \subset$$
$$\mathcal{R}\begin{pmatrix} Y(\Omega)\Phi(\Omega,\rho)B_{1,0}(\rho) & Y(\Omega)\Phi(\Omega,\rho)B_{1,1}(\rho) & \dots & Y(\Omega)\Phi(\Omega,\rho)B_{1,n-1}(\rho) \end{pmatrix}.$$

□

4. Optimal control

We consider the minimum energy problem, with the following performance index

$$J(u_1) = \int_0^\Omega \| u_1(s)\|^2_{U(s)} ds$$

with

$$u_1^*(s)U(s)u_1(s) = < u_1(s), U(s)u_1(s) > = \| u_1(s)\|^2_{U(s)}$$

and where $U \in L^\infty([0,\Theta], M_p(\mathbb{R}))$ is a positive definite symmetric matrix. Hence $U^{-1}(s)$ exists for all $s \in [0,\Theta]$. $< .,. >$ denotes the inner product in the space $\mathbb{R}^p$.
Let the matrix

$$\Delta(T) = \int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)U^{-1}(s)B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^* ds. \tag{4.1}$$

is invertible, and we have

$$\bar{u}_1(s) = U^{-1}(s)B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2), \qquad s \in [0,\Theta]. \tag{4.2}$$

We have the following result of optimal control.

Theorem 4.1 *Let $u_1(s)$ be any control such that*

$$Y(\Omega)H_1u_1 + Y(\Omega)H_2u_2 = 0.$$

Let $\bar{u}_1(s)$ be the control defined in (4.2), then the control $\bar{u}_1(s)$ satisfying

$$Y(\Omega)H_1\bar{u}_1 + Y(\Omega)H_2u_2 = 0,$$

and

$$\int_0^\Omega \| u_1(s)\|^2_{U(s)} ds \geq \int_0^\Omega \| \bar{u}_1(s)\|^2_{U(s)} ds, \tag{4.3}$$

and the minimum performance index is given by

$$J(\bar{u}_1) = \int_0^\Omega \| \bar{u}_1(s)\|^2_{U(s)} ds = \| (-Y(\Omega)H_2u_2)\|^2_{\Delta^{-1}(T)}.$$

Proof: Let $\bar{u}_1(s)$ be the control defined in (4.2), then

$$\begin{aligned} y(\Omega) &= Y(\Omega)\Phi(\Omega,0)z_0 \\ &+ \int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)U^{-1}(s)B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2)ds \\ &+ Y(\Omega)H_2u_2 \\ &= Y(\Omega)\Phi(\Omega,0)z_0. \end{aligned}$$

Hence we have the equality

$$\int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)\bar{u}_1(s)ds = \int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)u_1(s)ds,$$

we obtain

$$< \int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)(u_1(s)-\bar{u}_1(s))ds, \Delta^{-1}(T)(-Y(\Omega)H_2u_2) >= 0.$$

With the aid of (4.2), we have

$$\int_0^\Omega < u_1(s)-\bar{u}_1(s), \bar{u}_1(s) >ds = 0. \tag{4.4}$$

One has

$$\begin{aligned} \int_0^\Omega \| u_1(s)\|^2_{U(s)}ds &= \int_0^\Omega \| \bar{u}_1(s)\|^2_{U(s)}ds + \int_0^\Omega \| u_1(s)-\bar{u}_1(s)\|^2_{U(s)}ds \\ &+ 2\int_0^\Omega < u_1(s)-\bar{u}_1(s), \bar{u}_1(s) >ds. \end{aligned} \tag{4.5}$$

Using (4.4) and (4.5), we can verify that the inequality (4.3) holds.

The minimum performance index, is given by the equality

$$\begin{aligned} J(\bar{u}_1) &= \int_0^\Omega \| \bar{u}_1(s)\|^2_{U(s)}ds \\ &= \int_0^\Omega \| U^{-1}(s)B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2)\|^2_{\mathbb{R}^p}ds \\ &= \int_0^\Omega < U^{-1}(s)B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2), U(s)U^{-1}(s) \\ &\quad B_1(s)^*\Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2) >ds \end{aligned}$$

Since the matrix $\Delta(T)$ is symmetric, hence by the properties of the inner product, it is easy to verify that

$$\begin{aligned} J(\bar{u}_1) &= \int_0^\Omega < (-Y(\Omega)H_2u_2), \Delta^{-1}(T)Y(\Omega)\Phi(\Omega,s)B_1(s)U^{-1}(s)B_1(s)^* \\ & \quad \Phi(\Omega,s)^*Y(\Omega)^*\Delta^{-1}(T)(-Y(\Omega)H_2u_2) > ds \\ &= < (-Y(\Omega)H_2u_2), \Delta^{-1}(T)\int_0^\Omega Y(\Omega)\Phi(\Omega,s)B_1(s)U^{-1}(s)B_1(s)^* \\ & \quad \Phi(\Omega,s)^*Y(\Omega)^*ds\Delta^{-1}(T)(-Y(\Omega)H_2u_2) > \\ &= < (-Y(\Omega)H_2u_2), \Delta^{-1}(T)(-Y(\Omega)H_2u_2) > \\ &= \| (-Y(\Omega)H_2u_2)\|^2_{\Delta^{-1}(T)}. \end{aligned}$$

□

5. Minimum energy problem

In this section, we present a more general approach which consists to consider the domination problem as minimization one of a cost function defined on $L^2(0,\Omega;\mathbb{R}^p)$ as follows

$$\begin{aligned} J(u_1) &= < Q(Y(\Omega)H_1u_1 + Y(\Omega)H_2u_2), Y(\Omega)H_1u_1 + Y(\Omega)H_2u_2 > \\ &+ \int_0^\Omega < W(\rho)(Y(\rho)H_1(\rho)u_1 + Y(\rho)H_2(\rho)u_2), Y(\rho)H_1(\rho)u_1 + Y(\rho)H_2(\rho)u_2 > d\rho \\ &+ \int_0^\Omega < U(\rho)u_1(\rho), u_1(\rho) > d\rho \end{aligned}$$

where $Q \in M_n(\mathbb{R})$ and $W \in L^\infty([0,\Theta], M_n(\mathbb{R}))$ are positive symmetric matrixes, and $U \in L^\infty([0,\Theta], M_p(\mathbb{R}))$ is a positive definite symmetric matrix.. We have the following result.

Proposition 5.1

If $dJ(\bar{u}_1) = 0$ and $d^2J(\bar{u}_1)$ is a positive definite matrix, then $\bar{u}_1$ is a unique control $\bar{u}_1 \in L^2(0,\Omega;\mathbb{R}^p)$ such that

$$J(\bar{u}_1) = \inf_{u_1 \in L^2(0,\Omega;\mathbb{R}^p)} J(u_1)$$

with $\bar{u}_1$ given by

$$\begin{aligned} \bar{u}_1 &= -[(H_1)^*Y(\Omega)^*QY(\Omega)H_1 + (H_1(.))^*Y(.)^*W(.)Y(.)H_1(.) + U(t)]^{-1} \\ & \quad \times[(H_1)^*Y(\Omega)^*QY(\Omega)H_2 + (H_1(.))^*Y(.)^*W(.)Y(.)H_2(.)]u_2 \end{aligned}$$

Proof: We have

$$\begin{aligned}
J(u_1) &= < QY(\Omega)H_1u_1, Y(\Omega)H_1u_1 > +2 < QY(\Omega)H_1u_1, Y(\Omega)H_2u_2 > \\
&+ < QY(\Omega)H_2u_2, Y(\Omega)H_2u_2 > \\
&+ \int_0^\Omega < W(\rho)Y(\rho)H_1(\rho)u_1, Y(\rho)H_1(\rho)u_1 > d\rho + 2\int_0^\Omega < W(\rho)Y(\rho)H_1(\rho)u_1, Y(\rho)H_2(\rho)u_2 > d\rho \\
&+ \int_0^\Omega < Y(\rho)H_2(\rho)u_2, Y(\rho)H_2(\rho)u_2 > d\rho + \int_0^\Omega < U(\rho)u_1(\rho), u_1(\rho) > d\rho
\end{aligned}$$

and for $h \in L^2(0,\Omega;\mathbb{R}^p)$, we have

$$\begin{aligned}
\frac{1}{2}dJ(u_1)h &= < QY(\Omega)H_1u_1, Y(\Omega)H_1h > + < QY(\Omega)H_1h, Y(\Omega)H_2u_2 > \\
&+ \int_0^\Omega < W(\rho)Y(\rho)H_1(\rho)u_1, Y(\rho)H_1(\rho)h > d\rho + \int_0^\Omega < W(\rho)Y(\rho)H_1(\rho)h, Y(\rho)H_2(\rho)u_2 > d\rho \\
&+ \int_0^\Omega < U(\rho)u_1(\rho), h(\rho) > d\rho \\
&= < (H_1)^*Y(\Omega)^*QY(\Omega)H_1u_1, h > + < (H_1)^*Y(\Omega)^*QY(\Omega)H_2u_2, h > \\
&+ < (H_1(.))^*Y(.)^*W(.)Y(.)H_1(.)u_1, h > + < (H_1(.))^*Y(.)^*W(.)Y(.)H_2(.)u_2, h > \\
&+ < U(.)u_1, h >
\end{aligned}$$

We have

$$dJ(\bar{u}_1) = 0$$

i.e.

$$\begin{aligned}
&(H_1)^*Y(\Omega)^*QY(\Omega)H_1\bar{u}_1 + (H_1)^*Y(\Omega)^*QY(\Omega)H_2u_2 + \\
&(H_1(.))^*Y(.)^*W(.)Y(.)H_1(.)\bar{u}_1 + (H_1(.))^*Y(.)^*W(.)Y(.)H_2(.)u_2 \\
&+U(.)\bar{u}_1 = 0
\end{aligned} \tag{5.1}$$

and

$$d^2J(\bar{u}_1) = (H_1)^*Y(\Omega)^*QY(\Omega)H_1 + (H_1(.))^*Y(.)^*W(.)Y(.)H_1(.) + U(.)$$

is a positive definite matrix because $Q \in M_n(\mathbb{R})$ and $W \in L^\infty([0,\Theta], M_n(\mathbb{R}))$ are positive symmetric matrixes, and $U \in L^\infty([0,\Theta], M_p(\mathbb{R}))$ is a positive definite symmetric matrix.

By (5.1), one gets

$$\begin{aligned}
&[(H_1)^*Y(\Omega)^*QY(\Omega)H_1 + (H_1(.))^*Y(.)^*W(.)Y(.)H_1(.) + U(.)]\bar{u}_1 = \\
&-[(H_1)^*Y(\Omega)^*QY(\Omega)H_2 + (H_1(.))^*Y(.)^*W(.)Y(.)H_2(.)]u_2
\end{aligned}$$

then

$$\begin{aligned}
\bar{u}_1 &= -[(H_1)^*Y(\Omega)^*QY(\Omega)H_1 + (H_1(.))^*Y(.)^*W(.)Y(.)H_1(.) + U(.)]^{-1} \\
&\quad \times[(H_1)^*Y(\Omega)^*QY(\Omega)H_2 + (H_1(.))^*Y(.)^*W(.)Y(.)H_2(.)]u_2.
\end{aligned}$$

□

6. Conclusion

Domination in linear perturbed systems is a critical aspect of control theory. We study the notion of domination for linear time-varying systems. The objective is to compare or classify the input operators respecting the output equation. We show some conditions to characterise this concept. Then we study the optimal control which ensures the domination.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

Funding: No funds, grants, or other support was received.

Conflict of interest: It is declared that none of the authors have any competing interests in this manuscript.

References

1. Afifi L., Amimi N., El Jai A., Magri E. M, Regional domination in distributed systems. Applied Mathematical Sciences, 6(19):913-924, (2012).
2. Amissi C., Magri E. M., Lhous M., and Afifi L., Compensation Problem in Linear Fractional Order Disturbed Systems. Mathematical Modelling and Analysis. Volume 29, Issue 3, 546-559, 2024. https://doi.org/10.3846/mma.2024.18927, (2024).
3. Amissi C., Magri E. M., Lhous M., and Afifi L., Remediability problem in linear time-varying systems with disturbances. Bol. Soc. Paran. Mat. v. 2025 (43) : 1-19. ISSN-0037-8712. https://doi:10.5269/bspm.76256, (2025).
4. Chang A., An algebraic characterization of controllability. IEEE Trans. Automatic Control, AC-10, pp. 112-113, (1965).
5. Curtain R.F., Pritchard A.J., Infinite Dimensional Linear Systems Theory. Lecture Notes in Control and Information Sciences, vol. 8, Berlin, (1978).
6. Fenga, H. , Wu, X.H., and Guo, B.Z., Dynamics Compensation in Observation of Abstract Linear Systems. arXiv:2009.01643. https://doi.org/10.48550/arXiv.2009.01643. 3 Sep (2020).
7. Hizazi H., Amissi C., Lhous M., Magri E. M., Domination in linear fractional-order distributed systems. MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 2, pp. 455-462, (2024).
8. Huang, Y., and Xue, W., Active disturbance rejection control: Methodology and theoretical analysis, ISA transactions, 53:4, 963?976, (2014).
9. Larrache A., Lhous M., Ben Rhila S., Rachik M., Tridane A., An output sensitivity problem for a class of linear distributed systems with uncertain initial state. Archives of Control Sciences, 30(1): 77-93, (2020).
10. Magri E.M., Amissi C., Afifi L., Lhous M., On the minimum energy compensation for linear time-varying disturbed systems. Archives of Control Sciences, Volume 32(LXVIII), No. 4, pages 1-22, 2022. 10.24425/acs.2022.143669, (2022).
11. Mei, Q., She, J., Liu, Z. et al. Estimation and compensation of periodic disturbance using internal-model-based equivalent-input-disturbance approach. Sci. China Inf. Sci. 65, 182205, https://doi.org/10.1007/s11432-020-3192-5, (2022).
12. Rachik M., Lhous M., An Observer-based control of linear systems with uncertain parameters, Archives of Control Sciences, Volume 26(LXII), No. 4, pp. 565-576, (2016).
13. Souhail S., and Afifi L., Cheap controls for disturbances compensation in hyperbolic delayed systems. International Journal of Dynamical Systems and Differential Equations, 10:6, 511-536, (2020).
14. Yu, P., Wu, M., She, J.H., Robust tracking and disturbance rejection for linear uncertain system with unknown state delay and disturbance. IEEE/ASME Trans Mechatron, 23: 1445-1455, (2018).

AMISSI Chadi, MAGRI El Mostafa and LHOUS Mustapha,
Laboratory of Mathematical Analysis, Algebra and Applications,
Department of Mathematics and Computer Science,
Faculty of Sciences Ain Chock, Hassan II University of Casablanca,
Morocco.
E-mail address: `amissichadi@gmail.com, magrielmostafa1@gmail.com, mlhous17@gmail.com`