



Remediability Problem for a Semilinear Distributed Dynamical Systems

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ABSTRACT: In This paper we consider a semilinear distributed dynamical system. We focus our interest to the study of the remediability problem for a class of semi linear distributed system. The notion of both weak and exact remediability are described. We give some condition wich characterize the weak remediability of a case of parabolic system. To illustrate our work an example is given.

Keywords: Parabolic equation, semi linear systems, disturbance, compensation.

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1. Introduction

In system theory, controllability and observability form a pair central dual concepts that are closely dependent on the choice of input and output operators. A wealth of literature as devoted to the study of these concepts, offering various characterization of controllability and observability, both weak and exact as strategic actuation and measurement devices.

The detection of a disturbance is not generally sufficient to prevent the variance of a system but it is necessary to remedy to this perturbation. Serval works have focused on disturbance rejection [7] [6] and the notion of remediability is introduced and studied for a class of finals dimension linear and disturbed dynamical system [1] and also for discrete system [2] hyperbolic system [5] regional [4] and asymptotic case [3].

In this work, inspired by environmental issues, we study though the observation of a category of semi-linear systems subject to disturbances, the possibility of instigating in a finite time τ , the impact of disturbance (pollution, etc.) though observation with a given tolerance zone and the use of a suitably chosen control.

we generalize the studied of remediability for infinite dimensional semi-linear systems. We give some condition which ensure the weak and exact remediability, the existence of an input operator ensuring the compensation of disturbance. And it will be shown that the remediability is a weaker notion than the controllability of system.

2. Statement of the problem

A class of semi-linear distributed systems is modeled by the following equation:

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + F(v(t)) + \mathcal{B}u(t) + g(x, t) & \text{in } \Omega \times]0, \tau[\\ v(0) = v_0 \end{cases} \quad (2.1)$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and let V be a closed subspace of $L^2(\Omega)$. The spaces V and U are Hilbert spaces associated with the state and the control, respectively. for a given finale time $\tau > 0$, the

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state variable satisfies $v \in L^2(0, \tau, V)$, while the control u function belongs to $L^2(0, \tau; U)$. The operator \mathcal{A} is the generator of a strongly continuous semi-group (s.c.s.g.) on V , and the control operator \mathcal{B} is a bounded linear operator from U into V , that is, $\mathcal{B} \in \mathcal{L}(U, V)$. $\mathcal{L}(U, V)$ is the Banach space of bounded linear operators between these spaces. Let $F : V \rightarrow V$ be a real and globally Lipschitz function, i.e. $\exists \beta > 0$ such that

$$\|F(v_1) - F(v_2)\| \leq \beta \|v_1 - v_2\|, \forall v_1, v_2 \in V$$

The term g is a disturbance. The system (2.1) is augmented by the output equation

$$w(t) = \mathcal{C}v(t) \quad (2.2)$$

where $\mathcal{C} \in \mathcal{L}(V, W)$, W is the observation space (a Hilbert space).

The solution of (2.1) is noted $v_{u,g}$ and given by

$$v_{u,g}(t) = \mathcal{S}(t)v_0 + \int_0^t \mathcal{S}(t-s)F(v(s))ds + \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)ds + \int_0^t \mathcal{S}(t-s)g(s)ds.$$

We denote by $w_{u,g}$ the corresponding observation. If the system is only disturbed by a term g , the observation becomes

$$w_{0,g}(t) = \mathcal{S}(t)v_0 + \int_0^t \mathcal{S}(t-s)F(v(s))ds + \int_0^t \mathcal{C}\mathcal{S}(t-s)g(s)ds \neq \mathcal{C}\mathcal{S}(t)v_0$$

Then a control term Bu is subsequently introduced to compensate for the effect of the disturbance and to bring the observation back to its desired value at the final time τ , to the normal one, i.e.

$$w_{u,g}(T) = \mathcal{C}\mathcal{S}(T)v_0 + \int_0^T \mathcal{C}\mathcal{S}(T-s)F(v_{u,g}(s))ds + \int_0^T \mathcal{C}\mathcal{S}(T-s)\mathcal{B}u(s)ds + \int_0^T \mathcal{C}\mathcal{S}(T-s)g(s)ds$$

The following definitions follow.

Definition 2.1 (Exact remediability)

The system (2.1), augmented by (2.2), is said to be exactly remediable on $[0, \tau]$ if, for all disturbance $g \in L^2(0, \tau; V)$, there exists $u \in L^2(0, \tau; U)$ such that

$$w_{u,g}(\tau) = w_{0,0}(\tau) = \mathcal{C}\mathcal{S}v_0 + \int_0^\tau \mathcal{C}\mathcal{S}(\tau-s)F(v_{0,0}(s))ds$$

Definition 2.2 (Weak remediability)

The system (2.1), augmented by (2.2), is said to be weakly remediable on $[0, \tau]$ if for all disturbance $g \in L^2(0, \tau; V)$ and any $\varepsilon > 0$, there exists $u \in L^2(0, \tau; U)$ such that

$$\|w_{u,g}(\tau) - w_{0,0}(\tau)\| < \varepsilon$$

We examine hereafter the concept of weak remediability and then the exact one.

3. Weak remediability

This section addresses the approximate remediability of the evolutionary system (2.1) augmented by (2.2).

We define two operators G and R by :

$$\begin{aligned} G : L^2(0, \tau; U) &\longrightarrow V \\ u &\longrightarrow \int_0^\tau \mathcal{S}(\tau-s)\mathcal{B}u(s)ds \end{aligned} \quad (3.1)$$

$$\begin{aligned} R : L^2(0, \tau; V) &\longrightarrow W \\ g &\longrightarrow \int_0^\tau \mathcal{C}\mathcal{S}(\tau-s)g(s)ds \end{aligned} \quad (3.2)$$

And let H be the non linear operator defined for all $g \in L^2(0, \tau; V)$ as follows :

$$\begin{aligned} H : L^2(0, \tau; U) &\longrightarrow V \\ u &\longrightarrow \int_0^\tau \mathcal{S}(\tau - s)F(v_u(s))ds \end{aligned} \quad (3.3)$$

In the following result, we characterize the weak remediability of the linear system (corresponding to $F \equiv 0$)

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(t) + g(t) & \text{in } \Omega \times]0, \tau[\\ v(x, 0) = v_0 \end{cases} \quad (3.4)$$

augmented by (2.2)

Proposition 3.1 *The system (3.4), together with (2.2) is weakly remediable on $[0, \tau]$ if, and only if one of the following conditions is satisfied :*

1. $\overline{Im(\mathcal{C}G)} = \overline{Im(R)}$
2. $\ker(\mathcal{B}^*R^*) = \ker(R^*)$
3. $\mathcal{B}^*\mathcal{S}^*(.)\mathcal{C}^*w = 0 \implies \mathcal{S}^*(.)\mathcal{C}^*w = 0$
4. $\langle \mathcal{C}G\mathcal{G}^*\mathcal{C}^*w, w \rangle > 0$, for all $w \in W$ such that $R^*w \neq 0$
5. For all $g \in L^2(l)$ we have, $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I - \mathcal{C}G\mathcal{G}^*\mathcal{C}^*)^{-1}Rg = 0$

Proof:

The equivalence of 1. and 2. results from $\ker(\mathcal{B}^*R^*) = Im(\mathcal{C}G)^\perp = \overline{Im(\mathcal{C}G)}^\perp$ and $\ker(R^*) = Im(R)^\perp = \overline{Im(R)}^\perp$.

For the equivalence of 3. follows $R^* = \mathcal{S}^*(.)\mathcal{C}^*$ and $\ker(R^*) \subset \ker(\mathcal{B}^*R^*)$.

The equivalence of 3. and 4. results by contraposition.

Since $T = \mathcal{C}G\mathcal{G}^*\mathcal{C}^*$ is symmetric positive operator we conclude with theorem Lax-Milgram that $\alpha(\alpha I + \mathcal{C}G\mathcal{G}^*\mathcal{C}^*)$ is bijective for every $\alpha > 0$.

We now prove that $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + T)^{-1}w = P_{\ker T}y$ for all $w \in Y$.

Let $w \in W$, write $w = w_1 + w_2$ with $y_1 \in \ker T$ and $y_2 \in \overline{Im(T)}$

We have $w_1 = \alpha(\alpha I + T)^{-1}w_1 + (\alpha I + T)^{-1}Tw_1 = \alpha(\alpha I + T)^{-1}w_1$ and for $w_2 = Tx \in Im(T)$, so that $\alpha(\alpha I + T)^{-1}w_2 + T((\alpha I + T)^{-1}w_2 - x) = 0$

It follows that $\alpha(\alpha I + T)^{-1}w_2, (\alpha I + T)^{-1}w_2 - x \leq 0$ and thus by $\|(\alpha I + T)^{-1}w_2\| \leq \|x\|$. Consequently $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + T)^{-1}w_2 = 0$. By density, one can still prove that

$\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + T)^{-1}w_2 = 0$ for every $w_2 \in \overline{Im(T)}$

Then $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + T)^{-1}w = w_1 = P_{\ker(T)} w$

We may now return to the equivalence 5.

$\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + \mathcal{C}G\mathcal{G}^*\mathcal{C}^*)^{-1}Rg = 0 \iff P_{\ker T} Rg = 0 \iff Rg \in \overline{Im(T)} = \overline{Im(\mathcal{C}G)}$ □

Let us note the following

Remark 3.1 a) More generally, we can replace (5) in proposition by

$$\forall w \in \overline{Im(R)}, \lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + \mathcal{C}G\mathcal{G}^*\mathcal{C}^*)^{-1}w = 0$$

b) For $T = CGG^*C^*$ we have $T(\alpha I + T)^{-1} = I - \alpha(\alpha I + T)^{-1}$.

Then we obtain : $\lim_{\alpha \rightarrow 0^+} T(\alpha I + T)^{-1} = I - P_{\ker(T)} = P_{\overline{\text{Im}(T)}}$

We now consider the family of linear operators $\Gamma_\alpha : V \rightarrow L^2(0, \tau; U)$ for $0 < \alpha \leq 1$, defined by:

$$\Gamma_\alpha w = G^*C^*(\alpha I + CGG^*C^*)^{-1}w$$

is an approximate right inverse of the operator CG restricted to $\overline{\text{Im}(CG)}$, in the sense that

$$\lim_{\alpha \rightarrow 0^+} CG \Gamma_\alpha = P_{\overline{\text{Im}(CG)}}$$

in the strong topology.

we also have the following results.

Proposition 3.2 1. $\sup_{\alpha > 0} \|\alpha(\alpha I + CGG^*C^*)^{-1}\| \leq 1$

2. We have $CGu_\alpha = -Rg + \alpha(\alpha I + CGG^*C^*)^{-1}Rg$ For all $g \in L^2(0, \tau; V)$, where

$$u_\alpha = -G^*C^*(\alpha I + CGG^*C^*)^{-1}Rg, \quad \alpha \in]0, 1]$$

In the case of low compensation, we have $\lim_{\alpha \rightarrow 0^+} CGu_\alpha = -Rg$ and the error associated with this approximation denoted $E_\alpha Rg$, is given by the formula:

$$E_\alpha Rg = \alpha(\alpha I + CGG^*C^*)^{-1}Rg$$

In the following result and under the compacity condition on the semigroup $(\mathcal{S}(t))_{t \geq 0}$, we study the relationship between the weak remediability of the semi-linear system (2.1) augmented with (2.2), and that of the corresponding linear system (3.4) augmented with the output (2.2).

Theorem 3.1 Weak remediability of the system (3.4) with augmentation (2.2) on $[0, \tau]$, implies that the system (2.1) augmented by (2.2) is weakly remediable on the same interval.

Moreover, for all $g \in L^2(0, \tau; V)$, there exists a sequence of controls (u_{α_n}) which bring back the corresponding observation to an ε -neighborhood of $CS(\tau)v_0$ at time τ , the controls are given by

$$u_\alpha(t) = -\mathcal{B}^* \mathcal{S}(\tau - t)^* C^*(\alpha I + CGG^*C^*)^{-1}(Rg + CH(u_\alpha))$$

Proof: For each $g \in L^2(0, \tau; V)$ fixed.

We define the family of nonlinear operators $K_\alpha : L^2(0, \tau; U) \rightarrow L^2(0, \tau; U)$ for $(0 < \alpha \leq 1)$ given by

$$K_\alpha(u) = \Gamma_\alpha(-Rg - CH(u)) = -G^*C^*(\alpha I + CGG^*C^*)^{-1}(Rg + CH(u));$$

We first prove that for all $\alpha \in]0, 1]$ the operator K_α has a fixed point u_α .

Ended, since the semigroup $(\mathcal{S}(t))_t$ is compact, and using the smoothness and the boundedness of the non linear term f , it follows the operator H is compact and that $\overline{\text{Im}(H)}$ is relatively compact.

Consequently we have

$$H(u^1) - H(u^2) = \int_0^T \mathcal{S}(T-t)(F(v_{u^1}(t)) - f(v_{u^2}(t)))dt$$

$$\begin{aligned} \|H(u^1) - H(u^2)\| &\leq \int_0^T M \|F(v_{u^1}(t)) - f(v_{u^2}(t))\| dt \\ &\leq \int_0^T M\beta \|v_{u^1}(t) - v_{u^2}(t)\| dt \\ &\leq M\beta \int_0^T (\|G(u^1)(t) - G(u^2)(t)\| + \|H(u^1)(t) - H(u^2)(t)\|) dt \\ &\leq M\beta C_B \|u^1 - u^2\| + M\beta \int_0^T \|H(u^1)(t) - H(u^2)(t)\| dt \end{aligned}$$

For all T , then by the Gronwall lemma, we get

$$\| H(u^1) - H(u^2) \| \leq M\beta C_B e^{M\beta T} \| u^1 - u^2 \|$$

Therefore, H is a continuous operator and consequently K_α is also continuous. moreover, since f is bounded and $\| \mathcal{S}(t) \| \leq M_S e^{\omega t}$, there exists a constant $M > 0$ such that

$$\| H(u) \| \leq M, \quad \forall u \in L^2(0, T; U)$$

it Then follows that,

$$\| K_\alpha(u) \| \leq k = \| \Gamma_\alpha \| (\| Rg \| + M \| \mathcal{C} \|), \quad \forall u \in L^2(0, \tau; U).$$

Consequently, the operator K_α send the closed ball $B_r(0) \subset L_2(0, T; U)$ of center zero and radius $r \geq k$, into itself. by the Schauder fixed point Theorem it follows that K_α has a fixed point $u_\alpha \in B_r(0) \subset L_2(0, T; U)$.

Moreover, since the image of $\overline{Im(CH)}$ is compact, without loss of generality, we may assume that the sequence $CH(u_{\alpha_n})$ converges to $y \in \overline{Im(R)}$. So, if

$$u_{\alpha_n}(t) = -\Gamma_{\alpha_n}(Rg + \mathcal{C}Hu) = -G^* \mathcal{C}^* (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n}))$$

Then,

$$\begin{aligned} \mathcal{C}Gu_{\alpha_n} &= -\mathcal{C}GG^* \mathcal{C}^* (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) = \\ &= -(Rg + \mathcal{C}H(u_{\alpha_n})) - \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) \end{aligned}$$

Hence,

$$\mathcal{C}Gu_{\alpha_n} + \mathcal{C}H(u_{\alpha_n}) + Rg = -\alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n}))$$

To complete the proof , it is sufficient to prove that

$$\lim_{\alpha_n \rightarrow 0^+} -\alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) = 0$$

we have

$$\alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) = \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + w) + \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (\mathcal{C}H(u_{\alpha_n}) - w)$$

From proposition 3) we get that

$$\lim_{\alpha_n \rightarrow 0^+} \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) = \lim_{\alpha_n \rightarrow 0^+} \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (\mathcal{C}H(u_{\alpha_n}) - w)$$

On the other hand, $\| (\alpha I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (\mathcal{C}H(u_\alpha) - w) \| \leq \| \mathcal{C}H(u_\alpha) - w \|$

Therefore, since $\mathcal{C}H(u_{\alpha_n})$ converges to w , we get that

$$\lim_{n \rightarrow \infty} \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (\mathcal{C}H(u_{\alpha_n}) - w) = 0$$

Consequently,

$$\lim_{n \rightarrow \infty} \alpha_n (\alpha_n I + \mathcal{C}GG^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u_{\alpha_n})) = 0$$

So, we obtain the result:

$$\lim_{n \rightarrow \infty} \mathcal{C}Gu_{\alpha_n} + \mathcal{C}H(u_{\alpha_n}) + Rg = 0$$

□

4. Exact remediability

This section, we present a characterization of the exact remediability associated with the linear system (3.4) augmented by (2.2).

Proposition 4.1 *The system (3.4) augmented by (2.2) is exactly remediable on $[0, \tau]$ if and only if one of the following equivalent conditions is satisfied:*

1. $Im(\mathcal{C}G) = Im(R)$
2. There exists a constant $\rho > 0$ such that, for every $w \in W$, the following inequality holds:

$$\| \mathcal{S}^*(\cdot) \mathcal{C}^* w \|_{L^2(0, T; X)} \leq \rho \| \mathcal{B}^* \mathcal{S}^*(\cdot) \mathcal{C}^* w \|_{L^2(0, T; U)} \quad (4.1)$$

For the proof see [8], page 53.

Remark 4.1 In this case, the linear operator

$$\Gamma = G^* \mathcal{C}^* (\mathcal{C}G G^* \mathcal{C}^*)^{-1}$$

is a right inverse of the operator $\mathcal{C}G$ restricted to $Im(\mathcal{C}G) = Im(R)$.

Proposition 4.2 *If the system (3.4)+(2.2) is exactly remediable, then $\forall g \in L^2(0, \tau; V)$, the control $u = -G^* \mathcal{C}^* (\mathcal{C}G G^* \mathcal{C}^*)^{-1} Rg$ ensures the exact compensation of g .*

Theorem 4.1 *Exact remediability of the system (3.4) augmented by (2.2) on $[0, \tau]$ implies exact remediability of The system (2.1) augmented by (2.2) on the same time interval.*

Proof: For each fixed $g \in L^2(0, \tau; V)$, we consider the non linear operator $K : L^2(0, \tau; U) \rightarrow L^2(0, \tau; U)$ defined by

$$K(u) = \Gamma(-Rg - \mathcal{C}H(u)) = -G^* \mathcal{C}^* (\mathcal{C}G G^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u))$$

We first show that the operator K_α has a fixed point u_α .

Ended, since the semigroup $(\mathcal{S}(t))_t$ is compact, and taking into account the smoothness and the boundedness of the non linear term f , we obtain

$$\begin{aligned} \| H(u^1) - H(u^2) \| &\leq \int_0^\tau M \| F(v_{u^1}(t)) - F(v_{u^2}(t)) \| dt \\ &\leq \int_0^\tau ML \| v_{u^1}(t) - v_{u^2}(t) \| dt \\ &\leq ML \int_0^\tau (\| G(u^1)(t) - G(u^2)(t) \| + \| H(u^1)(t) - H(u^2)(t) \|) dt \end{aligned}$$

and

$$\begin{aligned} \| G(u^1)(t) - G(u^2)(t) \| &= \left\| \int_0^t \mathcal{S}(t-s) \mathcal{B} u(s) ds \right\| \leq M \| \mathcal{B} \| \int_0^t \| u^1(s) - u^2(s) \| dt \\ &\leq M \| \mathcal{B} \| \sqrt{t} \sqrt{\int_0^t \| u^1(s) - u^2(s) \|^2 dt} \\ &\leq M \| \mathcal{B} \| \sqrt{\tau} \sqrt{\int_0^\tau \| u^1(t) - u^2(t) \|^2 dt} \end{aligned}$$

then

$$\int_0^\tau \| G(u^1)(t) - G(u^2)(t) \| dt \leq \tau \sqrt{\tau} M \| \mathcal{B} \| \| u^1 - u^2 \|_{L^2(0, \tau; U)}$$

$$\| H(u^1) - H(u^2) \| \leq \tau \sqrt{\tau} M^2 \beta \| \mathcal{B} \| \| u - v \|_{L^2(0, \tau; U)} + M \beta \int_0^\tau \| H(u^1)(t) - H(u^2)(t) \| dt$$

the inequality of Gronwall, we have for $\tau_0 > 0$ and $T \in [0, \tau_0]$,

$$\| H(u^1) - H(u^2) \| \leq \tau_0 \sqrt{\tau_0} M^2 \beta \| \mathcal{B} \| \| u^1 - u^2 \| e^{M\beta T}$$

Then

$$\| H(u^1) - H(u^2) \| \leq \tau_0 \sqrt{\tau_0} e^{M\beta\tau_0} M^2 L \| \mathcal{B} \| \| u^1 - u^2 \|$$

we can choose τ_0 such that $\tau_0 \sqrt{\tau_0} e^{M\beta\tau_0} M^2 \beta \| \mathcal{B} \|$ as small we want since $G^* \mathcal{C}^* (\mathcal{C} G G^* \mathcal{C}^*)^{-1}$ is bounded, we can choose T_0 such that K is contracting. Then K admits a unique fixed-point $u \in L^2(0, T_0; U)$, and so

$$u = -G^* \mathcal{C}^* (\mathcal{C} G G^* \mathcal{C}^*)^{-1} (Rg + \mathcal{C}H(u))$$

assure the exactly compensation.

For any τ , we can make the compensation in $[\tau_0, 2\tau_0]$ with the initial value $\mathcal{S}(\tau_0)v_0$. then the compensation can be extended to interval $[0, n\tau_0]$ for all $n \in \mathbb{N}$, this completes the proof. \square

5. Case of a diffusion system

Consider an open and bounded domain Ω , included in \mathbb{R}^n , at the boundary $\Gamma = \partial\Omega$ assumet to be regular. this set Ω constitutes the physical domain of the system governed the equation (1). The state space considered $V = L^2(\Omega)$ has an orthonormal basis of eigenfunction φ_{n_j} for $n \geq 1$ and $j = 1, \dots, m_n$ of A , such that

$$\mathcal{A}\varphi_{n_j} = \mu_n \varphi_{n_j} \text{ for } j = 1, \dots, m_n, n \geq 1$$

such that $\mu_n \rightarrow -\infty$ as $n \rightarrow \infty$. The semigroup $(\mathcal{S}(t))_{t \geq 0}$ generated by \mathcal{A} is defined by

$$\mathcal{S}(t)v = \sum_{n \geq 1} e^{t\mu_n} \sum_{j=1}^{m_n} \langle v, \varphi_{n_j} \rangle \varphi_{n_j}$$

This assumption is satisfied when \mathcal{A} is a selfadjoint on V and has a compact resolvent $(sI - \mathcal{A})^{-1}$.

In the usual case of a system driven by p actuators $(\Omega_i, g_i)_{i=1,p}$ we set $U = \mathbb{R}^p$, and the operator \mathcal{B} is defined by

$$\begin{aligned} \mathcal{B} : \mathbb{R}^p &\longrightarrow L^2(\Omega) \\ u(t) &\longrightarrow \mathcal{B}u(t) = \sum_{i=1}^p b_i u_i(t) \end{aligned} \quad (5.1)$$

where $u = (u_1, \dots, u_p)^{tr} \in L^2(0, T; \mathbb{R}^p)$ and $b_i \in L^2(\Omega_i)$ with $\Omega_i = \text{supp}(b_i)$ for $i = 1, p$.

In general, the actuator supports satisfy $\Omega_i \cap \Omega_j = \emptyset$, for $i \neq j$.

The adjoint operator \mathcal{B}^* is given by:

$$\mathcal{B}^*v = (\langle b_1, v \rangle, \dots, \langle b_p, v \rangle)^T \text{ for } v \in V,$$

If the system output is provided by q sensors $(D_i, c_i)_{i=1,q}$ where $c_i \in V$, $D_i = \text{supp}(c_i)$ for $i = 1, q$ and $D_i \cap D_j = \emptyset$ for $i \neq j$ (Curtain and Pritchard, 1978; El Jai and Pritchard, 1988), then the observation operator \mathcal{C} is defined by

$$\begin{aligned} \mathcal{C} : L^2(\Omega) &\longrightarrow \mathbb{R}^q \\ v &\longrightarrow \mathcal{C}v = (\langle c_1, v \rangle_{L^2(\Omega)}, \dots, \langle c_q, v \rangle)^T \end{aligned} \quad (5.2)$$

Within this framework, the following characterization result holds.

Proposition 5.1 (3.4 – 2.2) *is weakly remediable on $[0, T]$ is equivalent to $\bigcap_{n \geq 1} \ker(M_n G_n^T) = \{0\}$ with*

$$M_n = (\langle b_i, \varphi_{n_j} \rangle)_{1 \leq i \leq p, 1 \leq j \leq r_n} \quad ; \quad G_n = (\langle c_i, \varphi_{n_j} \rangle)_{1 \leq i \leq q, 1 \leq j \leq r_n}$$

In particular, if there exists n_0 such that

$$\text{rank}(M_{n_0}G_{n_0}^\top) = q$$

Consequently, the nonlinear system (2.1) combined with (2.2) is weakly remediable on the interval $[0, T]$.

Example 5.1 In this example, we take the nonlinear term

$$f(v) = \arctan(v)$$

,which is globally lipschitz for the one-dimension case where $\Omega =]0, a[$ and the system (SNL) is described by the following equation.

$$(SNL) \begin{cases} \frac{\partial v}{\partial t}(x, t) = \frac{\partial^2 v}{\partial x^2}(x, t) + \arctan(v) + bu(t) + g(x) & \text{in } \Omega \times]0, T[\\ v(0, t) = v(a, t) = 0 & \text{on }]0, T[\\ v(x, 0) = 0 & \text{on }]0, a[\end{cases} \quad (5.3)$$

augmented by

$$\mathcal{C}v = \langle c, v \rangle_{L^2(\Omega)} \quad (5.4)$$

the associated linear is

$$(SL) \begin{cases} \frac{\partial v}{\partial t}(x, t) = \frac{\partial^2 v}{\partial x^2}(x, t) + bu(t) + g(t) & \text{in } \Omega \times]0, T[\\ v(0, t) = v(a, t) = 0 & \text{on }]0, T[\\ v(x, 0) = 0 & \text{on }]0, a[\end{cases} \quad (5.5)$$

We have $V = L^2(\Omega)$ and $\mathcal{A}v = \Delta v$ for $v \in D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$.

The operator \mathcal{A} admits a complete orthonormal family of eigenfunction $(\varphi_n)_n \in \mathbb{N}^*$ associated with the eigenvalues

$$\mu_n = -\frac{n^2\pi^2}{a^2}$$

the eigenfunction are explicity given by

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}.$$

The system (5.5) augmented by (5.4) is weakly remediable on $[0, T]$ if and only if there exists n_0 such that

$$\langle c, \varphi_{n_0} \rangle \langle b, \varphi_{n_0} \rangle \neq 0$$

Tn this case, we conclude with the theorem (4.1) that The system (5.3) augmented by (5.4) is weakly remediable on $[0, T]$

References

1. L. Affi, E. M. Magri and A. Jai, *Compensation problem in finite dimension linear dynamical systems*, Appl. Math. Sci. 2, 2219–2238, (2008).
2. L. Affi, M. Bahadi, A. El Jai and A. El Mizane, *The compensation problem in disturbed systems: asymptotic analysis, approximations and numerical simulations*, Int. J. Pure Appl. Math. 41, 927–956, (2007).
3. L. Affi, M. Bahadi, A. Chafiai and A. El Mizane, *Asymptotic compensation in discrete distributed systems: analysis, approximations and simulations*, Int. J. Appl. Math. Sci. 2, 99–137, (2008).
4. L. Affi, M. Bahadi and A. Chafiai, *A regional asymptotic analysis of the compensation problem in disturbed systems*, Appl. Math. Sci. 1, 2659–2686, (2007).
5. L. Affi, A. Chafiai and A. El Jai, *Sensors and actuators for compensation in hyperbolic systems*, Proc. MTNS, Perpignan, 19–23, (2000).
6. H. Leiva, N. Merentes, J. L. Sanchez and A. Tineo Moya, *Approximate controllability of semilinear non-autonomous systems in Hilbert space*, Adv. Dyn. Syst. Appl. 10, 57–75, (2015).
7. E. Zuazua, *Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities*, Control Cybernet. 28, 665–683, (1999).
8. R. F. Curtain and A. J. Pritchard, *Infinite dimensional linear systems theory*, Springer, Berlin, 10–50, (1978).

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