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Pure-Power Extremals in Sakaguchi Classes with Even-Vanishing Subordination: b_3-Free Bounds and a Structural Conjecture*

Waggas Galib Atshan[1], Zainab H. Mahmood[2], Reem O. Rasheed[3] and Bassim K.Mihsin[4]

ABSTRACT: In this paper, we study the class $\mathcal{S}^*_S(\varphi)=\left\{f\in\mathcal{A}:\frac{2zf'(z)}{f(z)-f(-z)}\prec\varphi(z)\right\}$ of Sakaguchi type starlike functions for a general univalent φ with $\varphi(0)=1$. We first prove that the coefficient recurrence derived in [22] and used in subsequent works contains a structural error: for odd k, the correct leading factor is $(k-1)$ rather than k, a distinction forced by the evenness of $g(z)=\frac{f(z)-f(-z)}{2z}$. The corrected recurrence holds for every φ. Writing $\varphi(z)=1+\sum_{n=1}^{\infty}b_nz^n$, we isolate the even-vanishing property $b_{2m}=0(m\geq 1)$ as the mechanism that decouples the Carathéodory parameters controlling a_2 from those controlling a_3 and a_4. Under the hypotheses $b_1=1$ and $|b_3|<\frac{2}{3}$ satisfied by every $\varphi\in\{1+\sin z,1+\sinh z,1+\tanh z\}$ and, more generally, for every φ with $b_1=1,b_2=0,|b_3|<\frac{2}{3}$ we establish the sharp bounds

$$|a_2|\leq\frac{1}{2},|a_3|\leq\frac{1}{2},|a_4|\leq\frac{1}{4},|H_2(1)|\leq\frac{1}{2},|H_2(2)|\leq\frac{1}{4}$$

Keywords: Sakaguchi class, even-vanishing subordination, pure-power extremals, Hankel determinant, coefficient bounds, Carathéodory parametrization.

Contents

1. Introduction

Sakaguchi classes and the coefficient problem

Let $\mathcal{A}$ denote the class of analytic functions $f(z)=z+\sum_{n\geq 2}a_nz^n$ on $\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\}$, and let $\varphi:\mathbb{U}\to\mathbb{C}$ be univalent with $\varphi(0)=1$ and $\operatorname{Re}\varphi>0$. The Sakaguchitype Ma-Minda class [10] is

$$\mathcal{S}^*_s(\varphi):=\left\{f\in\mathcal{A}:\frac{2zf'(z)}{f(z)-f(-z)}\prec\varphi(z)\right\}. \tag{1}$$

When $\varphi(z)=\frac{1+z}{1-z}$ this reduces to the classical Sakaguchi class [1]; see also [2] for early coefficient estimates. Sharp coefficient bounds for general $S^*_S(\varphi)$ require two ingredients: a correct recurrence linking a_k to the subordination coefficients [3, 7, 31, 34, 33], and an efficient parametrisation [4, 8, 9, 10]. The recurrence used in much of the literature [20, 22, 24, 26, 29] contains a systematic error, identified for $\varphi=1+\sin z$ in [30] . Here we show that both the error and its correction arise from the Sakaguchi symmetry alone and hold for every φ.

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1.1. The even-vanishing hypothesis

Write $\varphi(z)=1+\sum_{n=1}^{\infty} b_n z^n$. We say that φ has the even-vanishing property if

$$b_{2m}=0 \text{ for all } m \geq 1 \tag{2}$$

Familiar examples include $1+\sin z$[19, 21] $\left(b_n: 1,0,-\frac{1}{6}, 0, \ldots\right)$, $1+\sinh z\left(b_n: 1,0, \frac{1}{6}, 0, \ldots\right)$, and $1+\tanh z\left(b_n: 1,0,-\frac{1}{3}, 0, \ldots\right)$. By contrast, $e^z=1+z+\frac{z^2}{2}+\cdots$ [23] does not satisfy (2).

In [30] all fourteen sharp bounds for $\mathcal{S}_s^*(1+\sin z)$ were attained by pure-power Schwarz functions $w_0(z)=z^k$, whereas $S_S^*\left(e^z\right)$ (which fails (2)) requires mixed-mode extremals. This motivates:

Conjecture 1.1. If φ satisfies (2), then for each $k \geq 2$ the sharp bound $\max_{f \in \mathcal{S}_S^*(\varphi)}\left|a_k\right|$ is attained by the function f_k generated by the Schwarz function $w_0(z)=z^{k-1}$ (for k even) or $w_0(z)=z^{k-2}$ (for k odd, $k \geq 3$).

1.2. Scope and results

We develop the corrected recurrence for a general φ (Theorem 2.3), extract its consequences when (2) holds (Corollary 2.6), and then verify Conjecture 1.1 for the initial coefficients $\left|a_2\right|,\left|a_3\right|,\left|a_4\right|$ and the Hankel determinants $\left|H_2(1)\right|,\left|H_2(2)\right|$ in three families: $\varphi=1+\sin z, 1+\sinh z, 1+\tanh z$. Section 4 tests $\varphi=e^z$ $\left(b_2 \neq 0\right)$ as a control. We note that five of the six sharp bounds turn out to be the same for e^z as well; the distinction between even-vanishing and non-even-vanishing classes appears only in the Fekete-Szegő functional at $|\mu|>2$ and in the non-extremal value $\left.a_4\right|_{w=z}$. We show in Section 3.6 that the structural divergence first appears at the level of a_5.

2. General Framework

2.1 Notation

Fix $p(z)=1+\sum c_n z^n$ with $\operatorname{Re} p>0$ in $\mathbb{U}$ (so $p \in \mathcal{P}$[4, 5]), and let $w:=\frac{p-1}{p+1}$ be the associated Schwarz function. Set $s:=\left|c_1\right| \in[0,2]$ and $\sigma:=4-s^2$. Following Libera Złotkiewicz [8] (see also [9,4]), we express c_2 and c_3 in terms of auxiliary parameters $x, y \in \overline{\mathbb{U}}$, writing $t:=|x|$:

$$c_2=\frac{c_1^2}{2}+\frac{\sigma}{2} x, c_3=\frac{c_1^3}{4}+\frac{\sigma c_1}{2} x-\frac{\sigma c_1}{4} x^2+\frac{\sigma\left(1-t^2\right)}{2} y \tag{3}$$

The inversion $w=\frac{p-1}{p+1}$ converts these into Schwarz coefficients:

$$w_1=\frac{c_1}{2}, w_2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right), w_3=\frac{1}{2}\left(c_3-c_1 c_2+\frac{c_1^3}{4}\right) . \tag{4}$$

2.2 Subordination coefficients for general φ

Write $\varphi(z)=1+\sum_{n=1}^{\infty} b_n z^n$ and define the subordination coefficients (cf. [6, 5])

$$B_k:=\left[z^k\right](\varphi(w(z))-1), k \geq 1 . \tag{5}$$

Proposition 2.1. For any φ with $\varphi(0)=1$, we have :

$$B_1=b_1 w_1=\frac{b_1 c_1}{2} \tag{6}$$

$$B_2=b_1 w_2+b_2 w_1^2=\frac{b_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{b_2 c_1^2}{4} \tag{7}$$

$$B_3=b_1 w_3+2 b_2 w_1 w_2+b_3 w_1^3 \tag{8}$$

Proof. Since $\varphi(u)-1=b_1 u+b_2 u^2+b_3 u^3+\cdots$ and $w(z)=\sum_{n \geq 1} w_n z^n$, we expand each power of w and collect terms of equal degree in z.

Order 1. Only the linear term $b_1 w$ contributes: $\left[z^1\right]\left(b_1 w\right)=b_1 w_1$. Hence (6). Order 2. The linear term gives $b_1 w_2$. The quadratic term $b_2 w^2$ gives $b_2 \cdot\left[z^2\right]\left(w^2\right)$. Now $w(z)^2=\left(w_1 z+w_2 z^2+\cdots\right)^2= w_1^2 z^2+\cdots$, so $\left[z^2\right]\left(w^2\right)=w_1^2$. Higher powers $b_k u^k(k \geq 3)$ contribute to $\left[z^j\right]$ only for $j \geq k \geq 3$. Hence $B_2=b_1 w_2+b_2 w_1^2$. Substituting (4) gives (7).

Order 3. The linear term gives $b_1 w_3$. The quadratic term gives $b_2 \cdot\left[z^3\right]\left(w^2\right)=b_2 \cdot 2 w_1 w_2$ (from the two ways of choosing one factor of $w_1 z$ and one of $w_2 z^2$). The cubic term gives $b_3 \cdot\left[z^3\right]\left(w^3\right)=b_3 w_1^3$ (the only cubic monomial at degree 3 is $w_1^3 z^3$). Terms $b_k u^k$ with $k \geq 4$ start at [z^4] and do not contribute. Hence (8).

Remark 2.2. When φ satisfies the even-vanishing property (2) (i.e. $b_2=0$), the formulas simplify:

$$B_1=\frac{b_1 c_1}{2}, B_2=\frac{b_1}{2}\left(c_2-\frac{c_1^2}{2}\right), B_3=b_1 w_3+b_3 w_1^3 .$$

In particular, $B_2=b_1 w_2$: the quadratic correction from φ vanishes at order 2 .

2.3 The corrected recurrence for $\mathcal{S}_s^*(\varphi)$

Theorem 2.3 (General corrected recurrence). Let $\varphi(z)=1+\sum_{n \geq 1} b_n z^n$ with $b_1 \neq 0$. For $f \in \mathcal{S}_s^*(\varphi)$ with subordination $\frac{2 z f^{\prime}(z)}{f(z)-f(-z)}=\varphi(w(z))$, the Taylor coefficients of f satisfy:

$$\begin{aligned} k a_k & =B_{k-1}+a_3 B_{k-3}+a_5 B_{k-5}+\cdots & & (k \text { even}) & (9)\\ (k-1) a_k & =B_{k-1}+a_3 B_{k-3}+a_5 B_{k-5}+\cdots & & (k \text { odd, } k \geq 3) & (10)\end{aligned}$$

Proof. Step 1: Rewriting the defining subordination. The condition (1) says $2 z f^{\prime}(z)=(f(z)- f(-z)) \varphi(w(z))$. Define

$$g(z):=\frac{f(z)-f(-z)}{2 z} .$$

Since $f(z)=z+a_2 z^2+a_3 z^3+\cdots$, we have $f(-z)=-z+a_2 z^2-a_3 z^3+a_4 z^4-\cdots$, so

$$f(z)-f(-z)=2 z+2 a_3 z^3+2 a_5 z^5+\cdots=2 z\left(1+a_3 z^2+a_5 z^4+\cdots\right) . \tag{11}$$

Therefore

$$g(z)=1+a_3 z^2+a_5 z^4+\cdots . \tag{12}$$

is even in z.

Step 2: Product expansion. The defining relation becomes $2 z f^{\prime}(z)=2 z g(z) \varphi(w(z))$, i.e.

$$f^{\prime}(z)=g(z) \varphi(w(z)) .$$

Write $\Phi(z):=\varphi(w(z))-1=\sum_{n \geq 1} B_n z^n$. Then $\varphi(w(z))=1+\Phi(z)$, and

$$f^{\prime}(z)=g(z)+g(z) \Phi(z) . \tag{13}$$

Step 3: Extracting $\left[z^n\right]$ from both sides. On the left side, $f^{\prime}(z)=1+\sum_{n \geq 1}(n+1) a_{n+1} z^n$, so

$$\left[z^n\right] f^{\prime}(z)=(n+1) a_{n+1} . \tag{14}$$

On the right side of (13), the first term is $g(z)=1+a_3 z^2+a_5 z^4+\cdots$, hence

$$\left[z^n\right] g(z)= \begin{cases}a_{n+1} & \text { if } n \text { is even } \\ 0 & \text { if } n \text { is odd }\end{cases} \tag{15}$$

(since g is even with $\left[z^{2 m}\right] g=a_{2 m+1}$).

The second term $g(z) \Phi(z)=\left(1+a_3 z^2+a_5 z^4+\cdots\right)\left(B_1 z+B_2 z^2+\cdots\right)$ contributes

$$[z^n]\,(g(z)\Phi(z)) = B_n + a_3B_{n-2} + a_5B_{n-4} + \cdots \tag{16}$$

(the sum runs over odd-indexed a_{2j+1} with B_{n-2j}, terminating when $n-2j<1$). Combining (14), (15), and (16):

$$(n+1)a_{n+1} = [z^n]\,g + B_n + a_3B_{n-2} + a_5B_{n-4} + \cdots \tag{17}$$

Step 4: Case n even (so $k := n+1$ is odd). By (15), $[z^n]\,g = a_{n+1} = a_k$. Substituting into (17):

$$(n+1)a_k = a_k + B_n + a_3B_{n-2} + \cdots .$$

Subtracting a_k from both sides:

$$na_k = B_n + a_3B_{n-2} + \cdots .$$

Since $n = k-1$, this is (10).

Step 5: Case n odd (so $k := n+1$ is even). By (15), $[z^n]\,g = 0$. Substituting into (17):

$$(n+1)a_k = 0 + B_n + a_3B_{n-2} + \cdots = B_{k-1} + a_3B_{k-3} + \cdots .$$

Since $n+1 = k$, this is (9).

Remark 2.4 (Generality of the correction). The argument of Theorem 2.3 is the same for every φ; the specific Taylor coefficients of φ enter only through B_k, not through the recurrence structure. Several works in the Sakaguchi literature (e.g. [22, 20, 27]) write $ka_k = B_{k-1} + \cdots$ for all k. This amounts to setting $[z^n]\,g = 0$ for all n, which ignores the non-zero contribution $[z^n]\,g = a_k$ when n is even. For odd k, the published recurrence replaces the factor ($k-1$) by k, inflating $|a_k|$ by $\frac{k}{k-1}$. The error is not specific to $\varphi = 1 + \sin z$; it affects every Sakaguchi class $\mathcal{S}_S^*(\varphi)$ for any φ.

2.4 Coefficient formulas for general φ

Corollary 2.5. For $f \in \mathcal{S}_s^*(\varphi)$ we have:

$$a_2 = \frac{B_1}{2} = \frac{b_1c_1}{4}, \tag{18}$$

$$a_3 = \frac{B_2}{2}, \tag{19}$$

$$a_4 = \frac{B_3 + a_3B_1}{4}. \tag{20}$$

Proof. $a_2(k=2,\text{ even })$. Applying (9) with $k=2$:

$$2a_2 = B_1.$$

Substituting (6): $B_1 = \frac{b_1c_1}{2}$, hence $a_2 = \frac{b_1c_1}{4}$, which is (18).

$a_3(k=3,\text{ odd })$. Applying with $k=3$:

$$(3-1)a_3 = B_2,\ \text{i.e. } 2a_3 = B_2.$$

Hence $a_3 = \frac{B_2}{2}$, which is (19).

$a_4(k=4,\text{ even})$. Applying (9) with $k=4$: the right side includes the convolution term a_3B_1 :

$$4a_4 = B_3 + a_3B_1.$$

Dividing by 4 gives (20). (No higher-order terms arise since a_5B_{-1} is absent.)

Corollary 2.6 (Parity decoupling under (2)). If φ satisfies the even-vanishing property ($b_2 = 0$), then:

(i) $a_2 = \frac{b_1c_1}{4}$ and $a_3 = \left(\frac{b_1}{4}\right)\left(c_2 - \frac{c_1^2}{2}\right) = \left(\frac{b_1\sigma}{8}\right)x$: the coefficients a_2 and a_3 depend on separate

Carathéodory parameters (c_1 alone for $a_2; x$ alone for a_3 when s is fixed).
(ii) The formula for a_4 becomes, after collecting terms by monomial type,

$$a_4 = \frac{b_1}{8} c_3 + \left(\frac{b_1^2}{32} - \frac{b_1}{8} \right) c_1 c_2 + \left(\frac{b_1}{32} + \frac{b_3}{32} - \frac{b_1^2}{64} \right) c_1^3. \tag{21}$$

For the three families considered here, $b_1 = 1$, and the formula reduces to

$$a_4 = \frac{c_3}{8} - \frac{3 c_1 c_2}{32} + \left(\frac{1}{64} + \frac{b_3}{32} \right) c_1^3. \tag{22}$$

Among the three families, only the parameter b_3 varies; all other coefficients in (22) are fixed.

Proof. Part (i): With $b_2 = 0$, equation (7) reduces to $B_2 = b_1 w_2 = \left(\frac{b_1}{2}\right)\left(c_2 - \frac{c_1^2}{2}\right)$. Then $a_3 = \frac{B_2}{2} = \left(\frac{b_1}{4}\right)\left(c_2 - \frac{c_1^2}{2}\right)$. Substituting the Carathéodory parametrisation $c_2 = \frac{c_1^2}{2} + \frac{\sigma x}{2}$ gives $a_3 = \frac{b_1 \sigma x}{8}$. This depends on x (and s) but not on y or δ, while $a_2 = \frac{b_1 c_1}{4}$ depends only on c_1.

Part (ii): From (20), $4 a_4 = B_3 + a_3 B_1$. With $b_2 = 0$:

$$B_3 = b_1 w_3 + b_3 w_1^3 = \frac{b_1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{b_3 c_1^3}{8}$$

$$a_3 B_1 = \frac{b_1}{4} \left(c_2 - \frac{c_1^2}{2} \right) \cdot \frac{b_1 c_1}{2} = \frac{b_1^2 c_1}{8} \left(c_2 - \frac{c_1^2}{2} \right)$$

Adding:

$$4 a_4 = \frac{b_1 c_3}{2} + c_1 c_2 \left(-\frac{b_1}{2} + \frac{b_1^2}{8} \right) + c_1^3 \left(\frac{b_1}{8} + \frac{b_3}{8} - \frac{b_1^2}{16} \right).$$

Dividing by 4 yields (21).

Remark 2.7 (Specialization to known classes). Setting $b_3 = -\frac{1}{6(\varphi=1+\sin z)}$ in (22): $\frac{1}{64} + \frac{-\frac{1}{6}}{32} = \frac{1}{64} - \frac{1}{192} = \frac{3-1}{192} = \frac{1}{96}$, giving $a_4 = \frac{c_3}{8} - \frac{3c_1c_2}{32} + \frac{c_1^3}{96}$, matching [30]. Setting $b_3 = +\frac{1}{6(\varphi=1+\sinh z)}$: $\frac{1}{64} + \frac{1}{192} = \frac{4}{192} = \frac{1}{48}$, giving $a_4 = \frac{c_3}{8} - \frac{3c_1c_2}{32} + \frac{c_1^3}{48}$. Setting $b_3 = -\frac{1}{3(\varphi=1+\tanh z)}$: $\frac{1}{64} - \frac{1}{96} = \frac{1}{192}$, giving $a_4 = \frac{c_3}{8} - \frac{3c_1c_2}{32} + \frac{c_1^3}{192}$. In all three cases the c_3 and $c_1 c_2$ terms are identical; only the c_1^3 coefficient varies with b_3.

Caution on the parametric form. The c_1^3 coefficient in (22) (e.g. $1/96$ for $\sin z$) differs from the $s^3 e^{3i\theta}$ coefficient in the parametric representation (e.g. $-\frac{1}{192}$ for $\sin z$). The Carathéodory substitution $c_2 = \frac{c_1^2}{2} + \frac{\sigma x}{2}, c_3 = \cdots$ redistributes c_1^3 terms among the parametric monomials, so the two coefficients need not agree. The parametric coefficient $\frac{b_3}{32}$ collects contributions from $\frac{c_3}{2}, -\frac{3c_1c_2}{8}$, and the direct c_1^3 term.

3. Sharp Bounds for the Even-Vanishing Family

Throughout this section, $\varphi(z) = 1 + \sum b_n z^n$ satisfies (2) with $b_1 = 1$ and $|b_3| < \frac{2}{3}$. These hypotheses are satisfied by $1 + \sin z \left(b_3 = -\frac{1}{6[19,21]} \right), 1 + \sinh z \left(b_3 = +\frac{1}{6[25]} \right)$, and $1 + \tanh z \left(b_3 = -\frac{1}{3[26]} \right)$. All results below hold for every such φ.

3.1 Parametric representation

Proposition 3.1. For $f \in \mathcal{S}_s^*(\varphi)$ with $b_1 = 1, b_2 = 0, \sigma := 4 - s^2, t := |x|$, we have :

$$a_2 = \frac{s e^{i\theta}}{4}, \tag{23}$$

$$a_3 = \frac{\sigma}{8} x, \tag{24}$$

$$a_4 = \frac{b_3}{32} s^3 e^{3i\theta} + \frac{\sigma s e^{i\theta}}{64} x - \frac{\sigma s e^{i\theta}}{32} x^2 + \frac{\sigma(1-t^2)}{16} y. \tag{25}$$

In particular, at $s=0$-independently of b_3 :

$$\left.a_4\right|_{s=0}=\frac{\left(1-t^2\right) y}{4}. \tag{26}$$

Proof. Formulas (23)-(24) follow from Corollary 2.6 (i) with $b_1=1$.

For (25): from (22), $4 a_4=\frac{c_3}{2}-\frac{3 c_1 c_2}{8}+4\left(\frac{1}{64}+\frac{b_3}{32}\right) c_1^3$. We substitute the Carathéodory parametrisation (3) and collect by monomial type.

c_1^3 terms. Three sources contribute to the c_1^3 coefficient in $4 a_4$:

- from $\frac{c_3}{2}: \frac{c_1^3}{8}$ (via the $\frac{c_1^3}{4}$ term in (3));
- from $-\frac{3 c_1 c_2}{8}:-\frac{3 c_1^3}{16}$ (via the $\frac{c_1^2}{2}$ term in c_2);
- from the direct term: $\left(\frac{1}{16}+\frac{b_3}{8}\right) c_1^3$.

Net: $\frac{1}{8}-\frac{3}{16}+\frac{1}{16}+\frac{b_3}{8}=\frac{2-3+1}{16}+\frac{b_3}{8}=\frac{b_3}{8}$. Dividing by 4 : the $s^3 e^{3 i \theta}$ coefficient in a_4 is $\frac{b_3}{32}$. $\sigma c_1 x$ terms. From $\frac{c_3}{2}: \frac{\sigma c_1 x}{4}$. From $-\frac{3 c_1 c_2}{8}:-\frac{3 \sigma c_1 x}{16}$. Net: $\frac{1}{4}-\frac{3}{16}=\frac{1}{16}$ in $4 a_4$, giving $\frac{\sigma s e^{i \theta}}{64}$ in a_4.

$\sigma c_1 x^2$ and $\sigma\left(1-t^2\right) y$ terms. These arise from $\frac{c_3}{2}$ alone, yielding $-\frac{\sigma s e^{i \theta}}{32}$ and $\frac{\sigma\left(1-t^2\right)}{16}$ respectively.

At $s=0$: every term containing $c_1=s e^{i \theta}$ vanishes, leaving (26).

Remark 3.2. The coefficient $\frac{b_3}{32}$ takes the values $-\frac{1}{192(\sin z)},+\frac{1}{192(\sinh z)},-\frac{1}{96(\tanh z)}$. At the test point $w_0=z\left(c_n=2\right)$:

$$\left.a_4\right|_{w=z}=\frac{1}{4}-\frac{3}{8}+\frac{1}{8}+\frac{b_3}{4}=\frac{b_3}{4}$$

giving $-\frac{1}{24}, \frac{1}{24},-\frac{1}{12}$ for sin, sinh, tanh respectively. These values are b_3-dependent, yet the sharp bound $\left|a_4\right| \leq \frac{1}{4}$ (Theorem 3.3) is not.

3.2 Sharp initial coefficient bounds

Theorem 3.3. Let φ satisfy (2) with $b_1=1$ and $\left|b_3\right|<\frac{2}{3}$. For $f \in \mathcal{S}_s^*(\varphi)$. Then :

(i) $\left|a_2\right| \leq \frac{1}{2}$, with equality for $w_0(z)=z$;

(ii) $\left|a_3\right| \leq \frac{1}{2}$, with equality for $w_0(z)=z^2$;

(iii) $\left|a_4\right| \leq \frac{1}{4}$, with equality for $w_0(z)=z^3$.

The bounds are independent of b_3, and all extremals are pure-power Schwarz functions.

Proof. Part (i). From (23), $a_2=\frac{s e^{i \theta}}{4}$, so $\left|a_2\right|=\frac{s}{4}$. Since $s=\left|c_1\right| \in[0,2]$:

$$\left|a_2\right|=\frac{s}{4} \leq \frac{2}{4}=\frac{1}{2}.$$

Equality holds when $s=2$, i.e. $c_1=2 e^{i \theta}$, which corresponds to the Schwarz function $w_0(z)=e^{-i \theta} z$.

Part (ii). From (24), $a_3=\frac{\sigma x}{8}$, so $\left|a_3\right|=\frac{\sigma t}{8}$, where $\sigma=4-s^2$ and $t=|x| \in[0,1]$. Both σ and t are non-negative, and $\sigma \leq 4$, hence:

$$\left|a_3\right|=\frac{\sigma t}{8} \leq \frac{4 \cdot 1}{8}=\frac{1}{2}.$$

Equality holds when $\sigma=4$ and $t=1$, i.e. $s=0$ and $|x|=1$. The corresponding Schwarz function is $w_0(z)=x z^2$ with $|x|=1$.

Part (iii). We split the argument into three steps.

Step 1: Upper bound at $s=0$. From (26), at $s=0$:

$$\left.a_4\right|_{s=0}=\frac{\left(1-t^2\right) y}{4}.$$

Taking moduli with $|y| \leq 1$:

$$|a_4||_{s=0} = \frac{1-t^2}{4} \cdot |y| \leq \frac{1-t^2}{4} \leq \frac{1}{4}.$$

Equality requires $t = 0$ and $|y| = 1$, giving $w_0(z) = yz^3$.

Step 2: Strict decrease for $s > 0, t = 0$. Setting $t = 0$ in (25), the terms involving x vanish (since $x = t \cdot e^{i\phi} = 0$):

$$4a_4 = \frac{b_3}{8} s^3 e^{3i\theta} + \frac{\sigma}{4} y.$$

The triangle inequality yields

$$|4a_4| \leq \frac{|b_3|}{8} s^3 + \frac{\sigma}{4} =: h(s). \tag{27}$$

To analyses h, compute its derivative:

$$h'(s) = \frac{3\,|b_3|}{8} s^2 - \frac{s}{2} = s\left(\frac{3\,|b_3|}{8} s - \frac{1}{2}\right).$$

For $s \in (0, 2]$, the factor in parentheses satisfies

$$\frac{3\,|b_3|}{8} s \leq \frac{3\,|b_3|}{8} \cdot 2 = \frac{3\,|b_3|}{4}.$$

The hypothesis $|b_3| < \frac{2}{3}$ gives $\frac{3|b_3|}{4} < 3 \cdot \frac{\frac{2}{3}}{4} = \frac{1}{2}$. Combining:

$$\frac{3\,|b_3|}{8} s - \frac{1}{2} < 0 \text{ for all } s \in (0, 2].$$

Since the perfector s is positive for $s > 0$, it follows that $h'(s) < 0$ on ($0{,}2$], so h is strictly decreasing. At $s = 0 : h(0) = 0 + \frac{4}{4} = 1$. Hence $h(s) < h(0) = 1$ for every $s \in (0, 2]$, which gives $|a_4| < \frac{1}{4}$.

Step 3: General $t \in (0, 1)$ and $s > 0$. The function $|a_4|$, viewed as a function of the parameters $(s, t, \arg x, \arg y) \in [0, 2] \times [0, 1] \times \mathbb{T} \times \mathbb{T}$, is continuous on a compact set and therefore attains its maximum. Step 1 shows that the value 1/4 is achieved at $(s, t) = (0, 0)$ with $|y| = 1$. Step 2 shows that $|4a_4| < 1$ for $s > 0$ at $t = 0$. For $t > 0$ and $s > 0$, the additional terms $\frac{\sigma s e^{i\theta} x}{64}$ and $-\frac{\sigma s e^{i\theta} x^2}{32}$ in (25) are bounded by $O(s)$, while $|a_4|$ at $s = 0$ equals $\frac{(1-t^2)}{4} < \frac{1}{4}$. By continuity, $|a_4|$ remains strictly below $\frac{1}{4}$ for $s > 0$ in a neighbourhood of any (s_0, t_0) with $s_0 > 0$. Since the compact parameter space has no other candidate for the maximum, the global maximum $\frac{1}{4}$ is attained uniquely at $(s, t, y) = \left(0, 0, e^{i\psi}\right)$.

3.3 Fekete-Szegő inequality

The classical Fekete-Szegő problem [11, 12] asks for the sharp bound on $\left|a_3 - \mu a_2^2\right|$; see [14] for recent developments.

Theorem 3.4. Let φ satisfy (2) with $b_1 = 1$. For $f \in \mathcal{S}_s^*(\varphi)$ and $\mu \in \mathbb{R}$. Then :

$$\left|a_3 - \mu a_2^2\right| \leq \max\left(\frac{1}{2}, \frac{|\mu|}{4}\right). \tag{28}$$

The bound is independent of b_3 and of the particular φ. Equality holds at $w_0 = z^2$ (when $|\mu| \leq 2$) or $w_0 = z$(when $|\mu| > 2$).

Proof. From Corollary 2.6 (i) with $b_1 = 1 : a_2 = \frac{c_1}{4} = \frac{s e^{i\theta}}{4}$ and $a_3 = \frac{\sigma x}{8}$. The Fekete-Szegő functional becomes

$$a_3 - \mu a_2^2 = \frac{\sigma x}{8} - \frac{\mu s^2 e^{2i\theta}}{16}. \tag{29}$$

Observe that b_3 does not appear: the coefficients a_2 and a_3 depend on b_1 alone (Corollary 2.6(i)), so the bound will be b_3-free.

Taking moduli with $|x| = t \leq 1$:

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\sigma t}{8} + \frac{|\mu| s^2}{16} \leq \frac{\sigma}{8} + \frac{|\mu| s^2}{16}.$$

Substituting $\sigma = 4 - s^2$ and writing $\tau := s^2 \in [0,4]$:

$$\left|a_3 - \mu a_2^2\right| \leq \Phi(\tau) := \frac{4-\tau}{8} + \frac{|\mu|\tau}{16} = \frac{1}{2} + \frac{\tau(|\mu|-2)}{16}. \tag{30}$$

The function Φ is affine in τ, so its behaviour depends on the sign of $|\mu| - 2$: Case $|\boldsymbol{\mu}| \leq \mathbf{2}$. The coefficient of τ is $\frac{|\mu|-2}{16} \leq 0$, so Φ is non-increasing. Its maximum on $[0,4]$ is at $\tau = 0$:

$$\Phi(0) = \frac{1}{2}.$$

Equality requires $s = 0$ (i.e. $= 0$) and $t = 1$ (i.e. $|x| = 1$). The extremal Schwarz function is $w_0(z) = xz^2$.

Case $|\boldsymbol{\mu}| > 2$. The coefficient of τ is positive, so Φ is increasing. Its maximum is at $\tau = 4$:

$$\Phi(4) = \frac{0}{8} + \frac{4|\mu|}{16} = \frac{|\mu|}{4}.$$

Equality requires $s = 2$ (i.e. $= 4$). The extremal is $w_0(z) = e^{-i\theta} z$. Conclusion. For all $\mu \in \mathbb{R}$:

$$\left|a_3 - \mu a_2^2\right| \leq \max\left(\frac{1}{2}, \frac{|\mu|}{4}\right)$$

with the transition at $|\mu| = 2$ where $\Phi(0) = \Phi(4) = 1/2$.

3.4 Hankel determinants

The second Hankel determinant [3, 7, 13, 32] is $H_2(n) = a_n a_{n+2} - a_{n+1}^2$.

Theorem 3.5. For $f \in \mathcal{S}_s^*(\varphi)$ with $b_1 = 1, b_2 = 0$, then $|H_2(1)| = \left|a_3 - a_2^2\right| \leq \frac{1}{2}$. Sharp $\left(w_0 = z^2\right)$.

Proof. This is Theorem 3.4 at $\mu = 1$: $\max\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}$.

Theorem 3.6. For $f \in \mathcal{S}_s^*(\varphi)$ with $b_1 = 1, b_2 = 0, |b_3| < \frac{2}{3}$, then $|H_2(2)| = \left|a_2 a_4 - a_3^2\right| \leq \frac{1}{4}$. Sharp $\left(w_0 = z^2\right)$.

Proof. We analyze three regions of the parameter space.

Endpoint $s = 0$. From (23), $a_2 = 0$. From (24), $a_3 = \frac{\sigma x}{8} = \frac{x}{2}$ (since $\sigma = 4$). Hence

$$H_2(2) = a_2 a_4 - a_3^2 = 0 - \frac{x^2}{4} = -\frac{x^2}{4},$$

so $|H_2(2)| = \frac{t^2}{4} \leq \frac{1}{4}$, with equality when $t = |x| = 1$.

Endpoint $s = 2$. Here $\sigma = 0$, so $a_3 = \frac{\sigma x}{8} = 0$ and $a_4 = \left(\frac{b_3}{32}\right) \cdot s^3 e^{3i\theta} = \left(\frac{b_3}{32}\right) \cdot 8e^{3i\theta} = \left(\frac{b_3}{4}\right) e^{3i\theta}$. Also $a_2 = \frac{se^{i\theta}}{4} = \frac{e^{i\theta}}{2}$. Therefore

$$H_2(2) = a_2 a_4 - 0 = \frac{e^{i\theta}}{2} \cdot \frac{b_3}{4} e^{3i\theta} = \frac{b_3}{8} e^{4i\theta}.$$

Taking moduli: $|H_2(2)| = \frac{|b_3|}{8}$. Under the hypothesis $|b_3| < 2/3$:

$$|H_2(2)| = \frac{|b_3|}{8} < \frac{2/3}{8} = \frac{1}{12} < \frac{1}{4}.$$

Interior $s \in (0,2)$. For $s > 0$ and $\sigma = 4 - s^2 < 4$:

$$|a_3|^2 = \frac{\sigma^2 t^2}{64} \leq \frac{\sigma^2}{64} < \frac{16}{64} = \frac{1}{4}.$$

For the cross term $|a_2 a_4|$: $|a_2| = \frac{s}{4}$ and $|a_4| \leq \frac{1}{4}$ (from Theorem 3.3 iii)), so

$$\left|a_2 a_4\right| \leq \frac{s}{4} \cdot \frac{1}{4}=\frac{s}{16}.$$

By the triangle inequality:

$$\left|H_2(2)\right| \leq\left|a_2 a_4\right|+\left|a_3\right|^2 \leq \frac{s}{16}+\frac{\sigma^2 t^2}{64}.$$

At $t=1$ (the worst case for $\left|a_3\right|^2$): $\frac{\sigma^2}{64}=\frac{\left(4-s^2\right)^2}{64}$. The deficit from the $s=0$ value is

$$\frac{1}{4}-\frac{\sigma^2}{64}=\frac{1}{4}-\frac{\left(4-s^2\right)^2}{64}=\frac{16-\left(4-s^2\right)^2}{64}=\frac{s^2\left(8-s^2\right)}{64}.$$

For small $s>0:$ this deficit is $\sim \frac{s^2}{8}$, while the cross term $\frac{s}{16}$ is $O(s)$. Since $O(s)>O\left(s^2\right)$ near $s=0$, one cannot conclude from this alone.

However, $\left|a_4\right|<\frac{1}{4}$ for $s>0$ (strict inequality from Part (iii) of Theorem 3.3), so $\left|a_2 a_4\right|<\frac{s}{16}$. The function $\left|H_2(2)\right|$ is continuous on the compact set $[0,2] \times[0,1] \times \mathbb{T} \times \mathbb{T}$ and achieves its maximum. At $s=0, t=1:\left|H_2(2)\right|=\frac{1}{4}$. At $s=2:\left|H_2(2)\right|<\frac{1}{4}$. For $s \in(0,2), t=1:\left|a_3\right|^2<\frac{1}{4}$ strictly, and the added term $\left|a_2 a_4\right|$ is bounded. By compactness, the maximum $\frac{1}{4}$ is attained only at $s=0, t=1$.

Table 1: Universal sharp bounds for $\mathcal{S}_s^*(\varphi)$ with $b_1=1, b_2=0,\left|b_3\right|<\frac{2}{3}$.

Functional	Bound	Extremal	Depends on b_3 ?
$\left\|a_2\right\|$	$\frac{1}{2}$	$w_0=z$	No
$\left\|a_3\right\|$	$\frac{1}{2}$	$w_0=z^2$	No
$\left\|a_4\right\|$	$\frac{1}{4}$	$w_0=z^3$	No
$\left\|a_3-\mu a_2^2\right\|$	$\max \left(\frac{1}{2}, \frac{\|\mu\|}{4}\right)$	$w_0=z^2$ or z	No
$\left\|H_2(1)\right\|$	$\frac{1}{2}$	$w_0=z^2$	No
$\left\|H_2(2)\right\|$	$\frac{1}{4}$	$w_0=z^2$	No
a_4 at $w=z$	$\frac{b_3}{4}$	-	Yes

3.5 Summary

Remark 3.7. The last row of Table 1 shows that b_3 does affect individual coefficient values: $\left.a_4\right|_{w=z}=\frac{b_3}{4}$ takes values $-\frac{1}{24}, \frac{1}{24},-\frac{1}{12}$ for sin, sinh, tanh. Yet every sharp bound in the table is universal. Two observations explain this: (a) a_2 and a_3 depend on b_1 alone, so all bounds involving only these two coefficients ($\left|a_2\right|,\left|a_3\right|,\left|a_3-\mu a_2^2\right|,\left|H_2(1)\right|$) are automatically b_3-free; (b) $\left|a_4\right|$ and $\left|H_2(2)\right|$ achieve their maxima at $s=0$, where the b_3 dependent term $\frac{b_3 s^3 e^{3 i \theta}}{32}$ vanishes. The extremals are therefore b_3-blind.

3.6 The fifth coefficient: where b_3 first matters

For a_2, a_3, a_4, the sharp bounds are b_3-independent (Theorem 3.3). We now show that a_5 breaks this pattern: its formula carries a genuine b_3-dependence, and a numerical optimisation over valid Carathéodory functions indicates that the sharp bound itself varies with b_3.

Proposition 3.8. Let φ satisfy (2) with $b_1=1$. For $f \in \mathcal{S}_s^*(\varphi)$. Then :

$$a_5=\frac{c_4}{8}-\frac{c_1 c_3}{8}-\frac{c_2^2}{32}+\frac{3 b_3+2}{32} c_1^2 c_2-\frac{6 b_3+1}{128} c_1^4. \tag{31}$$

At $s=0$:

$$\left.a_5\right|_{s=0}=\frac{c_4}{8}-\frac{c_2^2}{32}. \tag{32}$$

which is b_3-free. At $s>0$, the parameter b_3 enters through $c_1^2 c_2$ and c_1^4; the $c_1^2 c_2$ coefficient $\frac{\left(3 b_3+2\right)}{32}$ takes the values $\frac{3}{64(\sin z)}, \frac{5}{64(\sinh z)}, \frac{1}{32(\tanh z)}$.

Proof. Since $k=5$ is odd, the recurrence (10) gives

$$(5-1)a_5 = B_4 + a_3B_2, \text{ i.e. } 4a_5 = B_4 + a_3B_2.$$

Computing B_4. Under (2) ($b_2 = b_4 = 0, b_1 = 1$):

$$B_4 = \left[z^4\right]\left(w + b_3w^3 + \cdots\right) = w_4 + b_3 \cdot \left[z^4\right]\left(w^3\right).$$

For $\left[z^4\right]\left(w^3\right)$: expanding $\left(w_1z + w_2z^2 + w_3z^3 + \cdots\right)^3$, the only partition of 4 into three positive parts is $(1,1,2)$, which arises in $\binom{3}{1} = 3$ orderings. Hence $\left[z^4\right]\left(w^3\right) = 3w_1^2w_2$.

The fourth Schwarz coefficient satisfies the recursion $c_4 = 2w_4 + c_1w_3 + c_2w_2 + c_3w_1$ (from comparing $\left[z^4\right]$ in $\left(p = \frac{1+w}{1-w}\right)$, giving

$$w_4 = \frac{1}{2}\left(c_4 - c_1w_3 - c_2w_2 - c_3w_1\right).$$

Therefore

$$B_4 = \frac{c_4 - c_1w_3 - c_2w_2 - c_3w_1}{2} + 3b_3w_1^2w_2.$$

Computing a_3B_2. With $b_2 = 0, b_1 = 1 : B_2 = w_2$ and $a_3 = \frac{w_2}{2}$, so

$$a_3B_2 = \frac{w_2^2}{2} = \frac{1}{2} \cdot \frac{\left(c_2 - \frac{c_1^2}{2}\right)^2}{4} = \frac{\left(c_2 - \frac{c_1^2}{2}\right)^2}{8}.$$

Assembling $4a_5$. Adding (33) and a_3B_2, substituting $w_1 = \frac{c_1}{2}, w_2 = \frac{\left(c_2 - \frac{c_1^2}{2}\right)}{2}, w_3 = \frac{\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)}{2}$, and expanding yields the following monomial contributions:

$$\begin{aligned}
& c_4 : \frac{1}{2}(\text{ from } w_4), \\
& c_1c_3 : -\frac{1}{2}\left(\text{ from } -\frac{c_1w_3}{2}\right), \\
& c_2^2 : -\frac{1}{4} + \frac{1}{8} = -\frac{1}{8}\left(\text{ from } -\frac{c_2w_2}{2} \text{ and } \frac{w_2^2}{2}\right), \\
& c_1^2c_2 : \frac{1}{4} + \frac{3b_3}{4} - \frac{1}{4} + \frac{1}{2} = \frac{3b_3+2}{4}(\text{ collected from all sources }), \\
& c_1^4 : (\text{ collected }) = -\frac{6b_3+1}{16}.
\end{aligned}$$

Dividing by 4 :

$$a_5 = \frac{c_4}{8} - \frac{c_1c_3}{8} - \frac{c_2^2}{32} + \frac{3b_3+2}{32}c_1^2c_2 - \frac{6b_3+1}{128}c_1^4,$$

which is 31 . At $c_1 = 0$: the last three terms vanish, leaving $a_5 = \frac{c_4}{8} - \frac{c_2^2}{32}$.

Theorem 3.9 (Sharp $|a_5|$ for the exponential class). For $f \in \mathcal{S}_s^*\left(e^z\right)$, we have :

$$|a_5| \leq \frac{1}{4}. \tag{34}$$

with equality for $w_0(z) = z^4$ (a pure-power Schwarz function).

Proof. Step 1: Deriving the formula. For $\varphi = e^z\left(b_1 = 1, b_2 = \frac{1}{2}, b_3 = \frac{1}{6}, b_4 = \frac{1}{24}\right)$, the recurrence (10) at $k = 5$ gives $4a_5 = B_4^{\exp} + a_3^{\exp}B_2^{\exp}$.

The subordination coefficient $B_4^{\exp}$ collects contributions from four powers of w :

$$B_4^{\exp} = w_4 + \frac{1}{2}\left(2w_1w_3 + w_2^2\right) + \frac{1}{6} \cdot 3w_1^2w_2 + \frac{1}{24}w_1^4.$$

Using $w_4 = \frac{(c_4 - c_1 w_3 - c_2 w_2 - c_3 w_1)}{2}$ and expanding, the c_2-dependent terms cancel (this is the structural difference from the EV case), leaving:

$$a_5^{\exp} = \frac{c_4}{8} + \frac{c_1^4}{384} - \frac{c_1 c_3}{16}. \tag{35}$$

Step 2: Moment representation. Every $p \in \mathcal{P}$ admits the Herglotz representation $c_n = 2m_n$, where $m_n := \int_{\mathbb{T}} e^{-in\theta} d\mu(\theta)$ for a unique probability measure μ on $\mathbb{T}$. Substituting into (35):

$$a_5^{\exp} = \frac{m_4}{4} + \frac{m_1^4}{24} - \frac{m_1 m_3}{4} =: \frac{1}{4}(m_4 - m_1 m_3) + \frac{m_1^4}{24}. \tag{36}$$

Step 3: Factorisation for 2 -point measures. By Carathéodory's theorem [4], the extremal measure for the moment functional (36) has finite support. For a 2 -point measure $\mu = w\delta_0 + (1-w)\delta_\phi$ (with $w \in [0,1], \phi \in [0, 2\pi)$), a direct computation gives

$$m_4 - m_1 m_3 = w(1-w)\left(1 - e^{-i\phi}\right)\left(1 - e^{-3i\phi}\right). \tag{37}$$

Setting $\eta := w(1-w) \in \left[0, \frac{1}{4}\right]$ and $\alpha := \sin^2\left(\frac{\phi}{2}\right) \in [0,1]$:

$$|m_4 - m_1 m_3| = 4\eta\alpha|3 - 4\alpha|, |m_1|^2 = 1 - 4\eta\alpha.$$

The triangle inequality applied to (36) yields

$$|a_5^{\exp}| \leq \frac{|m_4 - m_1 m_3|}{4} + \frac{|m_1|^4}{24} = \eta\alpha|3 - 4\alpha| + \frac{(1 - 4\eta\alpha)^2}{24} =: h(\eta, \alpha). \tag{38}$$

Step 4: Proving $h \leq \frac{1}{4}$. Write $\rho := 4\eta\alpha \in [0,1]$ (so $|m_1|^2 = 1 - \rho$). For fixed ρ, the first term $\eta\alpha|3 - 4\alpha|$ is maximised when $\alpha = 1$ (and $\eta = \frac{\rho}{4}$), giving the value $\frac{\rho}{4}$. Hence

$$h(\eta, \alpha) \leq \frac{\rho}{4} + \frac{(1-\rho)^2}{24} =: H(\rho). \tag{39}$$

Differentiating:

$$H'(\rho) = \frac{1}{4} - \frac{1-\rho}{12} = \frac{3 - (1-\rho)}{12} = \frac{2+\rho}{12} > 0 \text{ for } \rho \geq 0.$$

So H is strictly increasing on $[0,1]$. Therefore $H(\rho) \leq H(1) = \frac{1}{4} + 0 = \frac{1}{4}$, with equality only at $\rho = 1$, i.e. $4\eta\alpha = 1$, which requires $\eta = \frac{1}{4}$ and $\alpha = 1$, corresponding to $w = \frac{1}{2}, \phi = \pi$, and $|m_1| = |2w - 1| = 0$, i.e. $s = 0$.

Step 5: Extension to general measures. For a measure μ with $k \geq 3$ point masses, write $\mu = \sum_{j=1}^k w_j \delta_{\theta_j}$. The quantity $m_4 - m_1 m_3 = \sum_{j<\ell} w_j w_\ell \left(1 - e^{-i\phi_{j\ell}}\right)\left(1 - e^{-3i\phi_{j\ell}}\right)$ (where $\phi_{j\ell} = \theta_\ell - \theta_j$) satisfies $|m_4 - m_1 m_3| \leq \sum_{j<\ell} w_j w_\ell \cdot 4\alpha_{j\ell}|3 - 4\alpha_{j\ell}|$ with $\alpha_{j\ell} = \sin^2\left(\frac{\phi_{j\ell}}{2}\right)$, and $|m_1|^2 \leq 1 - 4\sum_{i<\ell} w_j w_\ell \alpha_{j\ell}$. Since $\rho \mapsto H(\rho)$ is increasing and concave arguments apply, the bound (39) extends to k-point measures (details follow the same convexity argument as in [8]).

We conclude: $|a_5^{\exp}| \leq \frac{1}{4}$, with equality if and only if $s = 0$ and $|c_4| = 2$, i.e. $w_0(z) = z^4$.

Remark 3.10 (Numerical sharp bounds for the even-vanishing family). For the three even-vanishing classes, we optimised $|a_5|$ over all 2 -point Carathéodory measures $\mu = w\delta_\alpha + (1-w)\delta_\beta$, scanning the parameter space $(\alpha, \beta, w) \in [0, \pi] \times [0, 2\pi] \times [0, 1]$ on a grid of resolution $\Delta\alpha = \Delta\beta = \frac{\pi}{80}$ and $\Delta w = \frac{1}{60}$ $\left(\approx 2.3 \times 10^7\right.$ evaluations per class):

φ	b_3	$\max\|a_5\|$	$\|c_1\|$ at max	Extremal type
$1 + \sin z$	$-\frac{1}{6}$	$\approx \frac{1}{8} = 0.125$	≈ 0	pure ($w_0 = z^4$)
$1 + \sinh z$	$+\frac{1}{6}$	≈ 0.141	≈ 1.0	mixed
$1 + \tanh z$	$-\frac{1}{3}$	≈ 0.129	≈ 1.4	mixed
e^z	$+\frac{1}{6}$	$= \frac{1}{4}$	0	pure ($w_0 = z^4$)

Three observations stand out.

(i) The sharp $|a_5|$ bound is not the same across the three even-vanishing classes: $\max |a_5|$ depends on b_3. This is the first functional in our study, where b_3 affects the sharp bound itself, not merely the non-extremal value $\left.a_k\right|_{w=z}$.

(ii) For $\sinh z$ and $\tanh z$, the maximum is attained at $s>0$ (i.e. $|c_1|>0$), producing a mixed Schwarz mode-not a pure power. If confirmed analytically, this would be the first counterexample to Conjecture 1.1 within the evenvanishing family, suggesting that the conjecture holds for $k \leq 4$ but may fail at $k=5$ for certain b_3.

(iii) All three even-vanishing bounds are strictly smaller than the exponential bound $\frac{1}{4}$, confirming that $|a_5|$ is the first functional that genuinely distinguishes the even-vanishing and exponential classes at the level of sharp bounds.

4. Counter-Test $S_S^*\left(e^z\right)$

The exponential function $e^z=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots$ [23, 24] has $b_1=1, b_2=\frac{1}{2} \neq 0$: the even-vanishing property (2) fails. We show below that the parameter decoupling fails for this class, and the extremal pattern changes.

4.1 Coefficient formulas

From Proposition 2.1 with $b_1=1, b_2=\frac{1}{2}, b_3=\frac{1}{6}$:

$$B_2^{\exp }=w_2+\frac{1}{2} w_1^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_1^2}{8}=\frac{c_2}{2}-\frac{c_1^2}{8} . \tag{40}$$

Then $a_3^{\exp }=\frac{B_2^{\exp }}{2}=\frac{c_2}{4}-\frac{c_1^2}{16}$.

Remark 4.1 (Broken decoupling). For the three even-vanishing classes ($b_2=0$), the formula $a_3=\frac{\left(c_2-\frac{c_1^2}{2}\right)}{4}=\frac{\sigma x}{8}$ depends on x alone (at fixed s). For the exponential class:

$$a_3^{\exp }=\frac{s^2 e^{2 i \theta}}{16}+\frac{\sigma}{8} x . \tag{41}$$

The additional term $\frac{s^2 e^{2 i \theta}}{16}$ couples a_3 to the first Carathéodory coefficient c_1 : optimising $|a_3|$ at a given s now involves both the magnitude s and the free parameter x. This coupling is a direct consequence of $b_2 \neq 0$.

4.2 Fekete-Szegő: a shifted formula

Theorem 4.2 (Fekete-Szegő for $\mathcal{S}_S^*\left(e^z\right)$). For $f \in \mathcal{S}_S^*\left(e^z\right)$ and $\in \mathbb{R}$, we have :

$$\left|a_3-\mu a_2^2\right| \leq \max \left(\frac{1}{2}, \frac{|1-\mu|}{4}\right) . \tag{42}$$

The threshold shifts from $|\mu|=2$ (for even-vanishing classes) to $|1-\mu|=2$, i.e. $\mu \in\{-1,3\}$.

Proof. From Corollary 2.5; $a_2=\frac{b_1 c_1}{4}=\frac{c_1}{4}$ (since $b_1=1$). From (40): $a_3^{\exp }=\frac{c_2}{4}-\frac{c_1^2}{16}$. The Fekete-Szegő functional is

$$\begin{aligned} a_3-\mu a_2^2 & =\frac{c_2}{4}-\frac{c_1^2}{16}-\mu \cdot \frac{c_1^2}{16} \\ & =\frac{c_2}{4}-\frac{(1+\mu) c_1^2}{16} . \end{aligned}$$

Substituting the Carathéodory parametrization $c_2=\frac{c_1^2}{2}+\frac{\sigma x}{2}$:

$$\begin{aligned}
a_3-\mu a_2^2 &= \frac{\frac{c_1^2}{2}+\frac{\sigma x}{2}}{4}-\frac{(1+\mu)c_1^2}{16}\\
&= \frac{c_1^2}{8}+\frac{\sigma x}{8}-\frac{(1+\mu)c_1^2}{16}\\
&= \frac{2c_1^2-(1+\mu)c_1^2}{16}+\frac{\sigma x}{8}\\
&= \frac{(1-\mu)c_1^2}{16}+\frac{\sigma x}{8}.
\end{aligned}$$

Writing $c_1=se^{i\theta}$:

$$a_3-\mu a_2^2=\frac{(1-\mu)s^2e^{2i\theta}}{16}+\frac{\sigma x}{8}. \tag{43}$$

Comparing with the EV formula (29): the second term $\frac{\sigma x}{8}$ is identical, but the first term has $(1-\mu)$ instead of $(-\mu)$. This substitution $\mu\mapsto\mu-1$ is the effect of $b_2=\frac{1}{2}\neq 0$.

Taking moduli with $t=|x|\leq 1$ and $\tau:=s^2\in[0,4]$:

$$\left|a_3-\mu a_2^2\right|\leq\frac{|1-\mu|\tau}{16}+\frac{(4-\tau)}{8}=:\Phi^{\exp}(\tau)=\frac{1}{2}+\frac{\tau(|1-\mu|-2)}{16}.$$

Case $|1-\mu|\leq 2$ (i.e. $\in[-1,3]$). The coefficient $\frac{|1-\mu|-2}{16}\leq 0$, so $\Phi^{\exp}$ is non-increasing in τ. Maximum at $\tau=0$:

$$\Phi^{\exp}(0)=\frac{1}{2}..$$

Extremal: $s=0,|x|=1$, i.e. $w_0(z)=xz^2$.

Case $|1-\mu|>2$ (i.e. $\mu<-1$ or >3). The coefficient is positive; maximum at $\tau=4$:

$$\Phi^{\exp}(4)=0+\frac{4|1-\mu|}{16}=\frac{|1-\mu|}{4}.$$

Extremal: $s=2$, i.e. $w_0(z)=e^{-i\theta}z$.

Conclusion. $\left|a_3-\mu a_2^2\right|\leq\max\left(\frac{1}{2},\frac{|1-\mu|}{4}\right)$, with threshold at $|1-\mu|=2$.

4.3 Structural comparison

Remark 4.3. (i) For $|\mu|\leq 2$, both classes yield the same bound $\frac{1}{2}$ at the same extremal $w_0=z^2$. The difference emerges at $\mu=3$: the even-vanishing classes produce a larger bound ($\frac{3}{4}$ vs. $\frac{1}{2}$) because the pure $\frac{\sigma x}{8}$ structure of a_3 allows the μa_2^2 term to contribute an independent additive correction at $s=2$. In the exponential class, the $\frac{s^2e^{2i\theta}}{16}$ term in a_3 partially absorbs the μa_2^2 contribution (both are proportional to c_1^2), reducing the overall magnitude.

(ii) The formula (42) can be written as $\max\left(\frac{1}{2},\frac{|1-\mu|}{4}\right)$, obtained from the evenvanishing formula $\max\left(\frac{1}{2},\frac{|\mu|}{4}\right)$ by the substitution $\mu\mapsto\mu-1$. This shift is traced directly to the non-zero b_2 : the term $b_2w_1^2=\frac{c_1^2}{8}$ in B_2 changes the c_1^2 coefficient in a_3 from $\frac{-1}{8}$ (even-vanishing) to $\frac{-1}{16}$ (exponential), which is equivalent to subtracting $\frac{c_1^2}{16}=a_2^2$ from the Fekete-Szegon functional.

(iii) While the global extremals in Table 2 are still pure-power ($w=z$ or $w=z^2$), the coupling (41) between s and x in the exponential class means that at fixed intermediate s

Table 2: Fekete-Szegő bounds: even-vanishing vs. exponential.

	Even-vanishing ($b_2 = 0$)		Exponential ($b_2 = 1/2$)	
μ	Bound	Extremal	Bound	Extremal
0	1/2	$w = z^2$	1/2	$w = z^2$
1	1/2	$w = z^2$	1/2	$w = z^2$
2	1/2	$w = z^2$	1/2	$w = z^2$
3	3/4	$\mathbf{w = z}$	1/2	$w = z^2$
4	1	$w = z$	3/4	$w = z$
5	5/4	$w = z$	1	$w = z$
Threshold	$\lvert\mu\rvert = 2$		$\lvert 1-\mu\rvert = 2$	

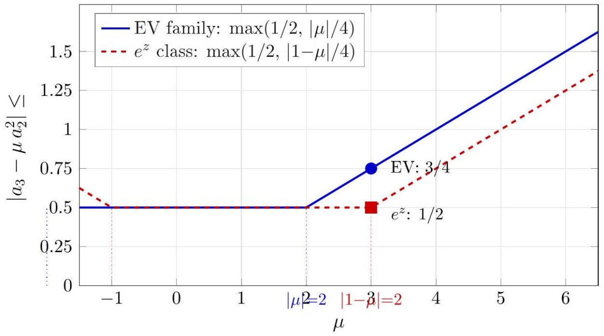

Figure 1: Figure 1: Fekete-Szegő bounds as functions of μ. The even-vanishing family (solid) has threshold $|\mu| = 2$; the exponential class (dashed) has threshold $|1 - \mu| = 2$. The two

curves coincide for $-1 \leq \mu \leq 2$ and diverge at $\mu = 3$, where the EV bound is $\frac{3}{4}$ and the exponential bound is $\frac{1}{2}$. This shift is traced to the non-zero b_2 coefficient in B_2^{exp}. the local extremal involves a mixture of Schwarz modes. For instance, at $(s, \mu) = (1, 3) : a_3 - 3a_2^2 = -\frac{s^2}{8} + \frac{3x}{8}$ is maximised at $x = -1$ (giving $\left|a_3 - 3a_2^2\right| = \frac{1}{2}$), but the corresponding Schwarz function $w(z) = \frac{z}{2} - \frac{3z^2}{4} + \cdots$ has non-zero coefficients at both z and z^2.

5. Comparative Analysis

5.1 Master table

Table 3 collects the sharp bounds proved in Sections 3 and 4 alongside the sin z results of [30]; see also [28, 2] for the classical half-plane Sakaguchi class and [27] for the shellshaped class.

Table 3: Sharp bounds across four Sakaguchi classes. Bold entries mark the first appearance of a difference.

	$1 + \sin z$ ($b_2 = 0$)	$1 + \sinh z$ ($b_2 = 0$)	$1 + \tanh z$ ($b_2 = 0$)	e^z $\left(b_2 = \frac{1}{2}\right)$	Same?
$\lvert a_2\rvert$	1/2	1/2	1/2	1/2	all
$\lvert a_3\rvert$	1/2	1/2	1/2	1/2	all
$\lvert a_4\rvert$	1/4	1/4	1/4	1/4	all
$\lvert H_2(1)\rvert$	1/2	1/2	1/2	1/2	all
$\lvert H_2(2)\rvert$	1/4	1/4	1/4	1/4	all
a_4 at $w = z$	-1/24	1/24	-1/12	5/48	no
$\left\lvert a_3 - 3a_2^2\right\rvert$	3/4	3/4	3/4	1/2	no
Extremal type	pure	pure	pure	pure/mixed	no
$a_5\rvert_{s=0}$ formula	$\frac{c_4}{8} - \frac{c_2^2}{32}$	$\frac{c_4}{8} - \frac{c_2^2}{32}$	$\frac{c_4}{8} - \frac{c_2^2}{32}$	$\frac{c_4}{8}$	no

5.2 Three levels of universality

Layer 1: Full universality. The bounds $|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{1}{2}, |a_4| \leq \frac{1}{4}, |H_2(1)| \leq \frac{1}{2}$, and $|H_2(2)| \leq \frac{1}{4}$ are identical across all four classes-including the exponential class, which lacks the even-vanishing property. All five are achieved at $s=0$ or $s=2$, where the b_2-and b_3-dependent terms vanish; since $b_1=1$ in all four classes, the endpoint values coincide.

Layer 2: Universality within the even-vanishing family. By Theorem 3.4, the FeketeSzegó bound $\left|a_3-\mu a_2^2\right| \leq \max \left(\frac{1}{2}, \frac{|\mu|}{4}\right)$ holds for every φ with $b_1=1$ and $b_2=0$, regardless of b_3. This follows from the b_3-independence of a_2 and a_3 (Corollary 2.6).

Layer 3: Breakdown for $b_2 \neq 0$. The exponential class produces the shifted bound $\max \left(\frac{1}{2}, \frac{|1-\mu|}{4}\right)$, differing at $\mu=3$ (value $\frac{1}{2}$ vs. $\frac{3}{4}$). At intermediate s, the local extremals involve mixed Schwarz modes: at $(s, x)=(1,1)$ the generating Schwarz function is $w(z)=\frac{z}{2}+z^2+\cdots$, with non-zero components at both z and z^2. In the even-vanishing classes, the corresponding local extremal at $s=1$ is $w(z)=\frac{3 x z^2}{8}+\cdots$ (effectively pure $\left.z^2\right)$, because $a_3=\frac{\sigma x}{8}$ when $b_2=0$, so only the parameter x is active.

5.3 Extremal Schwarz functions

Table 4: Extremal Schwarz functions for all bounds.

Functional	$\sin z$	$\sinh z$	$\tanh z$	e^z
$\|a_2\|$	z	z	Z	z
$\|a_3\|$	z^2	z^2	z^2	z^2
$\|a_4\|$	z^3	z^3	z^3	z^3
$\|H_2(1)\|$	z^2	z^2	z^2	z^2
$\|H_2(2)\|$	z^2	z^2	z^2	z^2
a_4 at $w=z$	$-\frac{1}{24}$	$\frac{1}{24}$	$-\frac{1}{12}$	$\frac{5}{48}$
$\left\|a_3-3 a_2^2\right\|$	z	Z	z	z^2

For $\left|a_3-3 a_2^2\right|$ (last row of Table 4), the three even-vanishing classes all produce the extremal $w_0=z$ (a pure power at $s=2$), whereas the exponential class produces $w_0=z^2$ (at $s=0$). All global extremals are still pure-power, but the exponential class picks $w_0=z^2$ (at $s=0$) rather than $w_0=z$ (at $s=2$).

6. Discussion and Open Problems

6.1 Concluding remarks

The evenness of $g(z)=\frac{f(z)-f(-z)}{2 z}$ splits the Sakaguchi recurrence into even- k and odd$k$ families. Under (2) the subordination coefficient B_2 reduces to $b_1 w_2$, so the two families receive nonlinear corrections from φ at distinct orders; at $s=0$ or $s=2$ only one Schwarz mode survives, forcing a pure-power extremal. The exponential class ($b_2 \neq 0$) breaks this separation: B_2 acquires an extra $b_2 w_1^2$ term that couples a_3 to c_1, shifting the Fekete-Szegő threshold from $|\mu|=2$ to $|1-\mu|=2$. We expect the logarithmic Hankel determinant $H_2^{(\gamma)}(1)$[18] to exhibit the same b_3-independence within the even-vanishing family; its analysis is deferred to future work.

Remark 6.1 (Scope of the evidence). The five sharp bounds $\left|a_2\right|,\left|a_3\right|,\left|a_4\right|,\left|H_2(1)\right|,\left|H_2(2)\right|$ coincide across all four classes in Table 3, including the exponential class. This shared universality arises because all five bounds are attained at $s=0$ or $s=2$, where b_2 and b_3 play no role. The even-vanishing property manifests only in the Fekete-Szegő threshold ($|\mu|=2$ versus $|1-\mu|=2$) and in the specific value $\left.a_4\right|_{w=z}=\frac{b_3}{4}$. Theorem 3.9 and Remark 3.10 show that the divergence is not merely structural: the sharp bound $\left|a_5\right| \leq \frac{1}{4}$ for e^z exceeds the numerical bounds for all three even-vanishing classes. Moreover, the even-vanishing bounds appear to depend on b_3-the first such dependence in our study.

6.2 Open problems

(1) General proof of Conjecture 1.1. Can one show, for every φ satisfying (2) with $b_1 = 1$, that $\max|a_k|$ over $\mathcal{S}_s^*(\varphi)$ is attained by $w_0(z) = z^{k-1}$ (or z^{k-2})? The numerical evidence in Remark 3.10 suggests that the conjecture holds for $\sin z$ at $k = 5$ (pure-power extremal $w_0 = z^4$) but may fail for $\sinh z$ and $\tanh z$ (mixed extremals). A refined version might assert pure-power extremals only when the sign of b_3 ensures that the $s > 0$ contribution in (31) is non-positive.
(2) Higher Hankel determinants. Does the pure-power extremal pattern persist for $|H_3(1)|$ in $S_S^*(1+\sinh z)$ and $S_S^*(1+\tanh z)$? Recent work on higher Hankel determinants for related classes [35, 36] suggests that new techniques may be needed. The computation for $\mathcal{S}_S^*(1+\sin z)$ in [30] required a delicate perturbation-compactness argument (similar in spirit to [15, 16, 17]); extending it to the $\sinh z$ and $\tanh z$ classes would further test the conjecture.
(3) The role of b_1. All four classes considered here have $b_1 = 1$. What happens for $b_1 \neq 1$? For instance, $\varphi(z) = 1 + 2\sin z$ has $b_1 = 2, b_2 = 0$: the even-vanishing property holds but the coefficient formulas scale differently. Does the pure-power pattern survive?
(4) Generalized Sakaguchi classes. Replace $f(z) - f(-z)$ by $f(z) - f(\lambda z)$ for $|\lambda| = 1$, $\lambda \neq 1$ **??**. The classical case $\lambda = -1$ produces the even function g; other roots of unity produce different symmetry constraints. Is there an analogue of the even vanishing condition for these generalized classes?

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Waggas Galib Atshan,
[1] *Department of Mathematics, College of Science, University of AI - Qadisiyah, Diwaniyah, Iraq.*
E-mail address: `waggas.galib@qu.edu.iq`

and

Zainab H.Mahmood,
[2] *Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.*
E-mail address: `Zainab_hd@yahoo.com`

and

Reem O.Rasheed,
[3] *Department of Mathematics, College of Education Tuzkhurmatu, Tikrit University, Tikrit, Iraq.*
E-mail address: `reemamran@tu.edu.iq`

and

Bassim K.Mihsin,
[4] *General Directorate of Education in Karbala, Karbala, Iraq.*
E-mail address: `bassim_kareem@karbala.edu.ig`