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Mathematical Modeling of Containing a Rumor via Counter-Rumor

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ABSTRACT: The spread of rumors, amplified by digital social networks, represents an increasing challenge for social cohesion, public health, and democratic stability. In this work, we propose a deterministic mathematical model of the epidemic type describing the coupled dynamics of a rumor and a counter-rumor within a homogeneous population. The model, structured into five compartments: ignorant, acceptor, spreader, counter-rumor spreaders, and educated allows us to analyze the conditions under which a rumor can persist or be eradicated. We derive the basic reproduction number $\Re_0$ and prove the existence of a rumor-free equilibrium. Using Lyapunov functions together with applying LaSalle's invariance principle, we establish the global asymptotic stability of this equilibrium under appropriate conditions on the model parameters. The model is then enhanced by incorporating three control policies, such as critical thinking education, awareness programs and legal action to mitigate the dissemination of rumor. Sensitivity analysis and parameter estimation are carried out using the least-squares method and the extended Kalman filter, based on data collected during Hurricane Harvey in 2017. These methods make it possible to calibrate the model and to carry out numerical simulations that clearly show how different control strategies influence the evolution of each compartment, as well as the variations observed on an hourly time scale. Overall, the results indicate that mathematical modeling provides a useful and reliable framework for supporting the design of intervention strategies aimed at limiting the spread of rumors in highly interconnected societies.

Keywords: Control strategies, mathematical modeling, equilibrium stability, reproduction number, rumor propagation, sensitivity analysis, estimation of parameters.

Contents

1. Introduction

Rumor propagation has become a recurring challenge in highly connected communication environments, where information can reach large audiences within minutes. In such settings, a rumor is not simply an informal exchange: it is a circulating piece of information that may affect social trust, individual decisions, and collective responses, especially when verification is delayed or absent. These features motivate the use of quantitative approaches that can describe how a rumor emerges, spreads, and eventually weakens, as well as how corrective information may change this trajectory.

Early mathematical models of rumor spreading were strongly influenced by epidemiology. A classical starting point is the stochastic framework introduced by Daley and Kendall, which partitions the population into ignorant, spreaders, and stiflers [8]. Soon after, Maki and Thompson proposed a deterministic formulation using ordinary differential equations, which became widely used because of its analytical accessibility [9]. For a long period, many studies relied on homogeneous and well-mixed assumptions [10]. The development of network science then led to an important extension of these ideas: Zanette investigated rumor diffusion on small-world networks and showed that structural randomness can substantially affect diffusion thresholds [11], while Moreno *et al.* studied scale-free networks and highlighted the amplifying role of heterogeneous connectivity patterns [12].

More recent work has aimed at improving behavioral realism and accounting for features observed in online platforms. For instance, the SIHR model proposed by Zhao *et al.* incorporates hesitation before transmission [13], and Huo *et al.* considered latent stages and optimal control components to represent institutional responses during media events [14]. In parallel, the growing availability of social-media data has enabled empirical calibration and validation efforts, including agent-based approaches that reproduce observed diffusion patterns and cascade effects [15]. Additional directions include rumor dynamics on multiplex structures and the inclusion of cognitive factors such as trust, emotions, and confirmation bias in mathematical formulations [16,17]. Taken together, these developments suggest that effective mitigation requires both mathematical structure and modeling choices that remain consistent with real communication constraints.

Within this perspective, the present work develops an extended compartmental model that explicitly incorporates the dynamics of counter-rumors. The key modeling choice is to treat counter-rumors as an *active corrective mechanism*, rather than as a passive termination process. This allows the model to capture not only the spread of misinformation but also the competitive influence of corrective content on the population states. The model is calibrated using real Twitter data collected during Hurricane Harvey (2017) [1]. Parameter estimation is performed using least squares and the extended Kalman filter, and a sensitivity analysis is used to examine the influence of parameters on the model outputs. Based on the calibrated setting, numerical simulations are carried out on an hourly time scale to quantify the effect of interventions on each compartment.

In addition, an optimal control framework is introduced to identify cost-effective strategies for limiting rumor propagation. The control design is motivated by standard optimal control methodology used in epidemiological and bio-mathematical systems [19,20,21,22], and it is adapted here to the context of rumor mitigation through (i) critical thinking education, (ii) awareness campaigns and corrective communication, and (iii) legal measures aimed at discouraging rumor spreading. This formulation provides a principled way to compare intervention intensities and their outcomes under explicit cost constraints.

The remainder of the paper is organized as follows. We first present the model and discuss basic properties such as positivity and boundedness of solutions. We then derive the threshold quantity and analyze the stability of the rumor-free equilibrium. Next, we report the sensitivity analysis and parameter estimation results based on Hurricane Harvey data [1]. Finally, we formulate and solve the optimal control problem and illustrate, through numerical simulations, how different intervention strategies influence rumor dynamics over time.

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2. The Mathematical Model Description

The model is composed of five compartments. I denotes the ignorant class, i.e., individuals who have not heard the rumor or seen it on social media. S denotes spreaders; this is the category of the population that spreads the rumor. A denotes the acceptors, representing individuals influenced by confirmation bias and does not check the truth of the information. C denotes counter-spreaders; this is the population that spreads the counter-rumor. Finally, E denotes educated individuals who are informed about the rumor and do not believe it.

$$\begin{cases} I' &= \mu N - \beta_1 \frac{IS}{N} - \beta_2 \frac{IC}{N} - \mu I \\ A' &= (1-\varepsilon_1)\beta_1 \frac{IS}{N} - \beta_3 \frac{AC}{N} - (\gamma + \mu) A \\ S' &= \varepsilon_1 \beta_1 \frac{IS}{N} + \gamma A - \mu S \\ C' &= \xi E - \mu C \\ E' &= \beta_2 \frac{IC}{N} + \beta_3 \frac{AC}{N} - (\xi + \mu) E \end{cases} \tag{2.1}$$

We assume that the total population is constant and $N = I(t) + S(t) + A(t) + C(t) + E(t)$

Remark 2.1 *It is presumed that all aforementioned parameters possess positive values.*

The parameters of the model are listed in Table 1 along with their biological significance.

Table 1: Parameter descriptions, values, sources, and units

Parameters	Description	Value	Source	Unit
μ	Natural death rate (per capita)	0.0082	[4]	
β_1	Rate at which Ignorant individuals believe the rumor	0.4269	fitted	hour^{-1}
β_2	Rate at which Ignorant individuals believe the counter-rumor	0.01	fitted	hour^{-1}
β_3	Rate at which Acceptors switch to believing the counter-rumor	0.4431	fitted	hour^{-1}
γ	Rate at which Acceptors become Spreaders	0.34	[1,2]	hour^{-1}
ξ	Rate at which Educated become counter-rumor spreaders	0.334	[1,2]	hour^{-1}
ε_1	Per-capita fraction of rumor believers who become Spreaders	0.7522	fitted	

3. Mathematical analysis of the model

3.1. Existence and Uniqueness of Solutions

In this section, we analyze the existence of a solution to the system of equations describing the model. The system is defined as follows:

$$\rho(X) = AX + B(X).$$

where X represents the vector of state variables $(I(t), A(t), S(t), C(t), E(t))$, and A and $B(X)$ are defined as:

$$X = \begin{bmatrix} I(t) \\ A(t) \\ S(t) \\ C(t) \\ E(t) \end{bmatrix}$$

$$\rho(X) = \begin{bmatrix} \dot{I}(t) \\ \dot{A}(t) \\ \dot{S}(t) \\ \dot{C}(t) \\ \dot{E}(t) \end{bmatrix}$$

The system matrix A and the nonlinear term $B(X)$ are given by:

$$\mathbf{A} = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\gamma+\mu) & 0 & 0 & 0 \\ 0 & \gamma & -\mu & 0 & 0 \\ 0 & 0 & 0 & -\mu & \xi \\ 0 & 0 & 0 & 0 & -(\xi+\mu) \end{pmatrix}$$

$$B(X) = \begin{bmatrix} \mu N - \beta_1 \frac{IS}{N} - \beta_2 \frac{IC}{N} \\ (1-\varepsilon_1)\beta_1 \frac{IS}{N} - \beta_3 \frac{AC}{N} \\ \varepsilon_1 \beta_1 \frac{IS}{N} \\ 0 \\ \beta_2 \frac{IC}{N} + \beta_3 \frac{AC}{N} \end{bmatrix}$$

Theorem 3.1 *The system* (2.1) *that fulfills the initial condition* $(I(0), A(0), S(0), C(0), E(0))$ *has a unique solution.*

Proof: In order to establish the existence and uniqueness of a solution to the system, we rewrite the system in the form

$$\rho(X) = AX + B(X).$$

To demonstrate the Lipschitz continuity of $B(X)$, we consider:

$$|B(X_1) - B(X_2)| \leq R|X_1 - X_2|.$$

where M is a constant derived from the system parameters. Specifically:

$$R = \max(2(R_2 + R_4), 2R_1, 2R_8, 2(R_3 + R_7)).$$

Where

$$R_1 = \frac{\beta_1}{N} \max\{I_1\}$$

$$R_2 = \frac{\beta_1}{N} \max\{S_2\}$$

$$R_3 = \frac{\beta_2}{N} \max\{I_1\}$$

$$R_4 = \frac{\beta_2}{N} \max\{C_2\}$$

$$R_5 = (1-\varepsilon_1)\frac{\beta_1}{N} \max\{I_1\}$$

$$R_6 = (1-\varepsilon_1)\frac{\beta_1}{N}\max\{S_2\}$$
$$R_7 = \frac{\beta_3}{N}\max\{A_1\}$$
$$R_8 = \frac{\beta_3}{N}\max\{C_2\}.$$

Hence, we obtain the following inequality for the system:

$$|\rho(X_1) - \rho(X_2)| \le T|X_1 - X_2|$$

where $T = \max\{R, ||A||\}$, which is finite. Thus, the function $\rho(t)$ is uniformly Lipschitz continuous. By the Picard-Lindelöf theorem, this guarantees the existence and uniqueness of the solution. [18]

□

3.2. Positivity of the solution

This section is devoted to studying the positivity and boundedness of the model's solutions for non-negative initial conditions.

Theorem 3.2 *If $I(0) \ge 0$, $A(0) \ge 0$, $S(0) \ge 0$, $C(0) \ge 0$ and $E(0) \ge 0$ then the solutions of system (2.1) remain nonnegative for all $t > 0$.*

Proof: From the first equation of system 2.1 we have:

$$\frac{dI}{dt} = \mu N - \beta_1 \frac{IS}{N} - \beta_2 \frac{IC}{N} - \mu I. \tag{3.1}$$

This can be rewritten as:

$$\frac{dI}{dt} = \mu N - M(t)I \qquad \text{Where} \qquad M(t) = \beta_1 \frac{IS}{N} + \beta_2 \frac{IC}{N} + \mu I. \tag{3.2}$$

We multiply equation 3.2 by $exp(\int_0^t M(s)ds)$, yielding:

$$\frac{dI}{dt} exp\Big(\int_0^t M(s)ds\Big) = (\mu N - M(t)I) exp\Big(\int_0^t M(s)ds\Big). \tag{3.3}$$

Thus

$$\frac{dI}{dt} exp\Big(\int_0^t M(s)ds\Big) + M(t)I(t) exp\Big(\int_0^t M(s)ds\Big) = \mu N\ exp\Big(\int_0^t M(s)ds\Big). \tag{3.4}$$

$$\frac{d}{dt}\Big(I(t) exp\Big(\int_0^t M(s)ds\Big)\Big) = \mu N\ exp\Big(\int_0^t M(s)ds\Big). \tag{3.5}$$

Taking integral with respect to q from 0 to t, we get:

$$I(t) exp\Big(\int_0^t M(s)ds\Big) - I(0) = \mu N \int_0^t \Big(exp\Big(\int_0^q M(s)ds\Big) dq\Big). \tag{3.6}$$

Multiplying the equation 3.6 by $exp(-\int_0^t M(s)ds)$, we obtain:

$$I(t) - I(0) exp\Big(-\int_0^t M(s)ds\Big) = \mu N\ exp\Big(-\int_0^t M(s)ds\Big) \int_0^t \Big(\int_0^q M(s)ds\Big) dq. \tag{3.7}$$

Thus

$$I(t) = I(0) exp\Big(-\int_0^t M(s)ds\Big) + \mu N\ exp\Big(-\int_0^t M(s)ds\Big) \int_0^t \Big(\int_0^q M(s)ds\Big) dq \ge 0. \tag{3.8}$$

Following the same procedure, it can be shown that $A(t) \ge 0$, $S(t) \ge 0$, $E(t) \ge 0$ and $C(t) \ge 0$ for all $t \ge 0$

□

3.3. Rumor-free equilibrium and the basic reproduction number:

In this subsection, we analyze the stability of the rumor-free equilibrium. The model has a rumor-free equilibrium given by $E^0 = (I^0, S^0, A^0, C^0, E^0) = (N, 0, 0, 0, 0)$.
To find $\Re_0$, we use the formula:

$$\Re_0 = \rho(FV^{-1}).$$

where ρ is the spectral radius. Here:

$$\mathbf{F} = \begin{bmatrix} \frac{(1-\varepsilon_1)\beta_1}{N} I^0 & 0 \\ \frac{\varepsilon_1\beta_1}{N} I^0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} (1-\varepsilon_1)\beta_1 & 0 \\ \varepsilon_1\beta_1 & 0 \end{bmatrix}$$

And $\mathbf{V}$ is given by:

$$\mathbf{V} = \begin{bmatrix} \mu & -\gamma \\ 0 & \gamma+\mu \end{bmatrix}$$

$$\mathbf{FV^{-1}} = \begin{bmatrix} \frac{(1-\varepsilon_1)\beta_1}{\mu} & \frac{(1-\varepsilon_1)\beta_1\gamma}{\mu(\gamma+\mu)} \\ \frac{\varepsilon_1\beta_1}{\mu} & \frac{\varepsilon_1\beta_1\gamma}{\mu(\gamma+\mu)} \end{bmatrix}$$

To find the spectral radius ρ , we have the characteristic polynomial

$$\begin{aligned} \det(FV^{-1} - \lambda I_2) = 0 &\Rightarrow \lambda^2 - [\frac{\varepsilon_1\beta_1}{\mu} + \frac{(1-\varepsilon_1)\beta_1\gamma}{\mu(\gamma+\mu)}]\lambda = 0. \\ &\Rightarrow \lambda[\lambda - \frac{\varepsilon_1\beta_1}{\mu} - \frac{(1-\varepsilon_1)\beta_1\gamma}{\mu(\gamma+\mu)}] = 0. \\ &\Rightarrow \lambda_1 = 0 \quad \text{or} \quad \lambda_2 = \frac{\varepsilon_1\beta_1}{\mu} + \frac{(1-\varepsilon_1)\beta_1\gamma}{\mu(\gamma+\mu)} \end{aligned}$$

The dominant eigenvalue of (FV^{-1}) is:

$$\lambda_2 = \frac{(\varepsilon_1\mu + \gamma)\beta_1}{\mu(\gamma+\mu)}.$$

Therefore, the reproduction number is given by:

$$\Re_0 = \frac{(\varepsilon_1\mu + \gamma)\beta_1}{\mu(\gamma+\mu)}.$$

The dynamics of the system depend crucially on the parameter $\varepsilon_1 \in [0, 1]$, which represents per capita (percentage) of believers of the rumor that become spreaders. To facilitate the analytical treatment of the model and to better understand its limiting behaviors, we consider two particular boundary scenarios:

- $\varepsilon_1 = 0$: No believer becomes a spreader.
- $\varepsilon_1 = 1$: All believers become spreaders.

These limiting situations provide a clear framework for studying the impact of the parameter ε_1 and for explicitly describing equilibria and their stability.

Case $\varepsilon_1 = 0$. When $\varepsilon_1 = 0$, the system (2.1) reduces to:

$$\begin{cases} I' &= \mu N - \beta_1 \dfrac{IS}{N} - \beta_2 \dfrac{IC}{N} - \mu I \\ A' &= \beta_1 \dfrac{IS}{N} - \beta_3 \dfrac{AC}{N} - (\gamma + \mu) A \\ S' &= \gamma A - \mu S \\ C' &= \xi E - \mu C \\ E' &= \beta_2 \dfrac{IC}{N} + \beta_3 \dfrac{AC}{N} - (\xi + \mu) E \end{cases} \tag{3.9}$$

The RFE is given by

$$E^0 = (N, 0, 0, 0, 0),$$

and the basic reproduction number is

$$\Re_0 = \frac{\gamma \beta_1}{\mu(\gamma + \mu)}.$$

Case $\varepsilon_1 = 1$. When $\varepsilon_1 = 1$, the system (2.1) reduces to:

$$\begin{cases} I' &= \mu N - \beta_1 \dfrac{IS}{N} - \beta_2 \dfrac{IC}{N} - \mu I \\ A' &= -\beta_3 \dfrac{AC}{N} - (\gamma + \mu) A \\ S' &= \beta_1 \dfrac{IS}{N} + \gamma A - \mu S \\ C' &= \xi E - \mu C \\ E' &= \beta_2 \dfrac{IC}{N} + \beta_3 \dfrac{AC}{N} - (\xi + \mu) E \end{cases} \tag{3.10}$$

The RFE is given by

$$E^0 = (N, 0, 0, 0, 0),$$

and the basic reproduction number is

$$\Re_0 = \frac{\beta_1}{\mu}.$$

4. Stability

4.1. Stability of the Rumor Free Equilibrium point

4.1.1. Local stability.

Theorem 4.1 *The rumor-free equilibrium (RFE) of the rumor model is locally asymptotically stable (LAS) if the following conditions are satisfied:*

$(C1)$ $\Re_0 < 1$.

$(C2)$ $\dfrac{\beta_2 \xi}{\mu(\xi + \mu)} < 1$.

And it is unstable if $\Re_0 > 1$.

Proof: The Jacobian of the system for E^0 is

$$\mathbf{J}(\mathbf{E^0}) = \begin{bmatrix} -\mu & 0 & -\beta_1 & -\beta_2 & 0 \\ 0 & -(\gamma+\mu) & (1-\epsilon_1)\beta_1 & 0 & 0 \\ 0 & \gamma & \epsilon_1\beta_1 - \mu & 0 & 0 \\ 0 & 0 & 0 & -\mu & \xi \\ 0 & 0 & 0 & \beta_2 & -(\xi+\mu) \end{bmatrix}.$$

$$\det(\mathbf{J}(\mathbf{E^0}) - \lambda \mathbf{I_5}) = (-\mu - \lambda))\det(M)\det(N).$$

Such that

$$\det(M) = \begin{vmatrix} -(\gamma+\mu) - \lambda & (1-\varepsilon_1)\beta_1 \\ \gamma & \varepsilon_1\beta_1 - \mu - \lambda \end{vmatrix} = \lambda^2 - (\varepsilon_1\beta_1 - 2\mu - \gamma)\lambda - (1-\varepsilon_1)\beta_1\gamma - (\varepsilon_1\beta_1 - \mu)(\gamma+\mu).$$

$$\Delta_M = (\varepsilon_1\beta_1 + \gamma)^2 + 4(1-\varepsilon_1)\beta_1\gamma > 0.$$

Then

$$\lambda_2, \lambda_3 = \frac{(\varepsilon_1\beta_1 - 2\mu - \gamma) \pm \sqrt{\Delta_M}}{2}$$

$$\det(N) = \begin{vmatrix} -\mu - \lambda & \xi \\ \beta_2 & -(\xi+\mu) - \lambda \end{vmatrix} = \lambda^2 + (\xi + 2\mu)\lambda + \mu(\xi+\mu) - \beta_2\xi.$$

$$\Delta_N = \xi^2 + 4\beta_2\xi > 0.$$

Then

$$\lambda_4, \lambda_5 = \frac{-(\xi + 2\mu) \pm \sqrt{\Delta_N}}{2}$$

And

$$\lambda_1 = -\mu$$

Under conditions (C1) and (C2), all eigenvalues of the Jacobian are negative. We conclude that the rumor-free equilibrium is locally asymptotically stable. □

4.1.2. Global Stability Analysis of Rumor-Free Equilibrium. In order to establish the global asymptotic stability of the system (2.1), we use Lyapunov function theory applied to the Rumor-free equilibrium.

Theorem 4.2 *The system (2.1) at the free equilibrium* $E^0 = (N, 0, 0, 0, 0)$ *is globally asymptotically stable if* $\Re_0 < 1$ *and unstable otherwise.*

Proof: The following Lyapunov function is introduced:

$$V : \Delta \to \mathbb{R}$$

$$V(I, A, S, C, E) = I - N - N \ln \frac{I}{N} + A + \frac{\gamma+\mu}{\gamma} S + \frac{\xi+\mu}{\xi} C + E$$

Where

$$\Delta = \{(I, A, S, C, E) \in \Omega / I > 0, A > 0, S > 0, C > 0, E > 0\}$$

The time derivative of the Lyapunov function is given by:

$$\begin{aligned} \frac{dV(I, A, S, C, E)}{dt} &= -\frac{\mu(I-N)^2}{I} - \frac{\beta_1 S(I-N)}{N} - \frac{\beta_2 C(I-N)}{N} + \frac{(1-\varepsilon_1)\beta_1 IS}{N} + \frac{\gamma+\mu}{\gamma}\varepsilon_1\beta_1\frac{IS}{N} \\ &\quad - \frac{\mu(\gamma+\mu)}{\gamma} S - \frac{\mu(\xi+\mu)}{\xi} C + \beta_2 \frac{IC}{N}. \\ &\le \frac{S}{\gamma}[\beta_1(\gamma + \varepsilon_1\mu) - \mu(\gamma+\mu)] + C[\beta_2 - \frac{\mu(\xi+\mu)}{\xi}] \end{aligned}$$

for

$$\Re_0 < 1 \quad and \quad \frac{\beta_2 \xi}{\mu(\xi+\mu)} < 1.$$

We have

$$\dot{V}(I, A, S, C, E) \leq 0.$$

Note that $\frac{dV(I,A,S,C,E)}{dt} = 0$ if and only if $I = N,\ A = 0,\ S = 0,\ C = 0,\ and\ E = 0$. Hence, by LaSalle's invariance principle, E^0 is globally asymptotically stable. □

5. Real Rumor Data and Parameter Estimation

5.1. Description of Data

This study is based on system (2.1) and uses real data from Twitter collected during Hurricane Harvey in 2017 [1]. The dataset used begins at 9 a.m. on August 23, 2017. On that date, a rumor circulated on Twitter claiming that individuals' immigration status had to be verified before they could be admitted to emergency shelters. This rumor spread rapidly across the platform. Subsequently, official sources issued communications refuting and clarifying this misinformation. These official denials began around 11 a.m. on August 25, 2017.

5.2. Known Model Parameters and Initial Conditions

In the first stage of the model parameter estimation process, and in order to ensure consistency with the available data, we assumed that the total population size was fixed at $N = 2774000$ [1,2], this number corresponds to the estimated number of daily active Twitter users in the United States and does not include bots. Based on available data, the mortality rate in the United States in 2017 is estimated at 0.0082. Based on this information, as well as other model adjustments, we set the initial conditions of the system (2.1) as follows: $I(0) = \frac{N-1}{N}$, $A(0) = 1$, $S(0) = 2$, $C(0) = 0$ and $E(0) = 1000$.

5.3. Parameters estimation strategy

The model was fitted to real data collected from Twitter during Hurricane Harvey in 2017, using two estimation approaches. The first is based on the least squares method, while the second uses the extended Kalman filter. The results of these adjustments are presented in the corresponding figure.1.

5.3.1. Least squares method (LS). Least squares is one of the best approaches to fit the model to observed real Twitter data of the spreader individuals [3]. This approach determines the best-fitting curve for the spreader individuals by minimizing the sum of squared residuals between the observed number of spreaders and the model predictions. Given the observations $\{y_t\}_{t=1}^{T}$ and model predictions $\hat{y}_t(\theta)$, the least-squares estimator solves

$$\hat{\theta} = \arg\min_{\theta} \ J(\theta), \qquad J(\theta) = \sum_{t=1}^{T} w_t \left[y_t - \hat{y}_t(\theta)\right]^2.$$

5.3.2. Extended Kalman Filter (EKF). An EKF is useful for the estimation of nonlinear dynamical systems [6,7]. Here we treat the real spreaders of the rumor counts as a second-order latent process:

$$\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_{t-1} + \mathbf{w}_t, \qquad \mathbf{F}_t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \tag{5.1}$$

$$\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{v}_t, \qquad \mathbf{H}_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{5.2}$$

where $\mathbf{x}_t = \begin{bmatrix} y_t & \dot{y}_t \end{bmatrix}^\top$ denotes an unobserved state vector, in which y_t represents the number of cases at time t, while $\dot{y}_t$ corresponds to its temporal variation rate. The stochastic disturbances affecting the system dynamics and the observations are modeled by the noise terms $\mathbf{w}_t \sim \mathcal{N}(0, \mathbf{Q})$ and $\mathbf{v}_t \sim$

$\mathcal{N}(0, \mathbf{R})$, respectively. Both noise components are assumed to follow zero-mean Gaussian distributions, with covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ selected through a tuning procedure:

$$\mathbf{Q} = \begin{bmatrix} 2.0 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad \mathbf{R} = [4] \tag{5.3}$$

The recursive filter alternates between prediction and correction steps to estimate the latent trajectory $\mathbf{x}_t$ over time.

Figure 1 illustrates the difference in fit between the two estimation methods applied to the observed dataset, meaning that the Least squares approach provides a better fitting to the observed Rumor spreader data compared to the EKF. The fitted parameter values are shown in Table 1 .

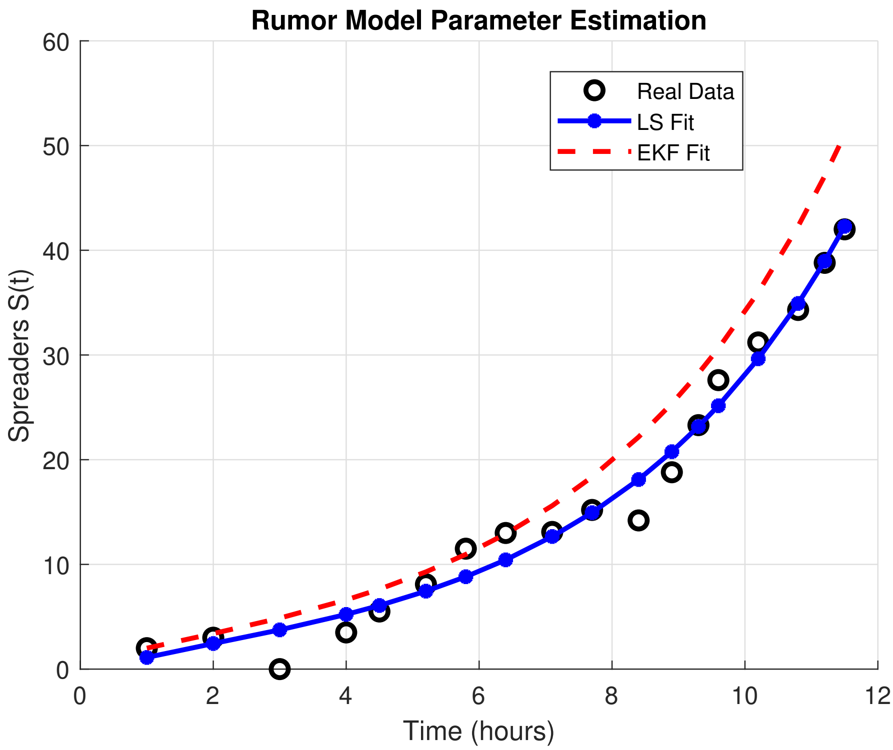

Figure 1: LS and EKF model fit to spreaders Rumor data in Hurricane Harvey in 2017.

6. Sensitivity analysis

In this section, we conduct the sensitivity analysis of the Rumor model (2.1) with respect to the model parameters. For this purpose, the sensitivity index was analyzed for each parameter. Sensitivity index lies between -1 and 1. Values close to 1 indicate a strong positive correlation, whereas values close to -1 correspond to a strong negative correlation. Value near zero suggests weak or negligible sensitivity. Positive index values reflect a direct relationship, while negative values correspond to an inverse relationship. The magnitude of sensitivity index is visually illustrated by the bar heights in the sensitivity figures. The parameters bounds considered in the sensitivity analysis are presented in table . We concentrate on analyzing the model parameter sensitivity to $\Re_0$. The major purpose is to find out how the parameters affect the population to spread a false rumor. This may be determined using the sensitivity index, which is defined as given in [5] by

$$S_p^{\Re_0} = \frac{p}{\Re_0} \frac{\partial \Re_0}{\partial p}.$$

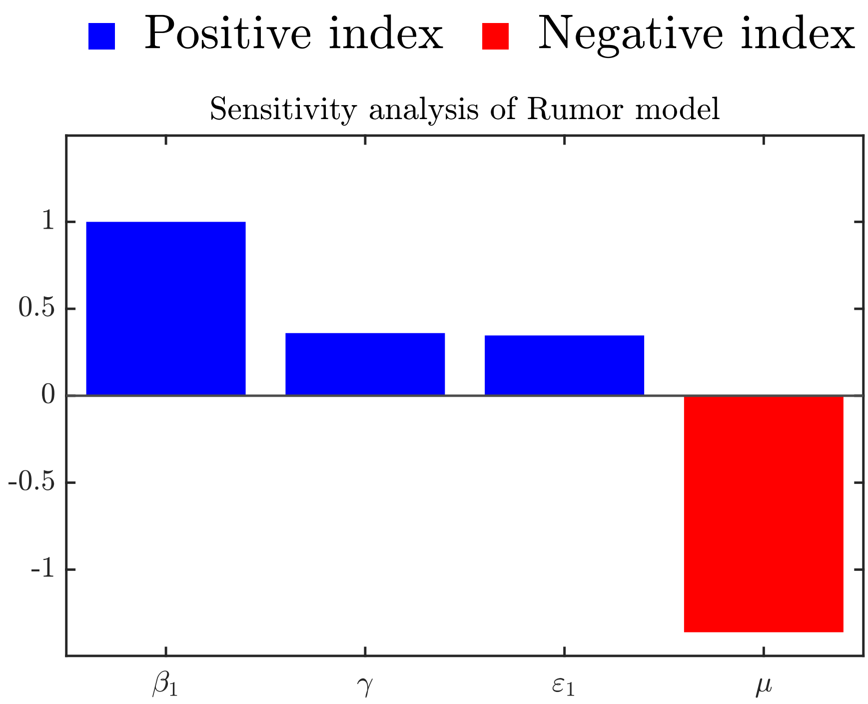

Figure 2: Sensitivity analysis of Rumor

Figure 2 presents the sensitivity analysis of Rumor reproduction $\Re_0$. The results indicate that the reproduction number is positively correlated when the ignorant individual believes the rumor and the progression rate when acceptors become spreaders and by the fraction of rumor when believers become spreaders. Among these , β_1 has an index value close to 1 , indicating a strong positive correlation, while γ and ϵ_1 show a positive effect of relatively low magnitude. This suggests that an increase (or decrease) in β_1 leads to a corresponding increase (or decrease) in $\Re_0$ the same applies to γ and ϵ_1 but with a smaller magnitude. In contrast the natural death rate μ is negatively correlated . The parameters bounds considered in the sensitivity analysis are presented in table .

Table 2: Sensitivity bounds of parameters and sensitivity indices of $\mathcal{R}_0$.

Parameter	**Fitted value**	**Lower bound**	**Upper bound**	**Sensitivity index**
β_1	0.2690	0.2018	0.3363	1.0000
β_2	0.0100	0.0075	0.0125	0.0000
β_3	0.4431	0.3323	0.5539	0.0000
γ	0.3400	0.2550	0.4250	0.3602
ξ	0.3340	0.2505	0.4175	0.0000
ε_1	0.2200	0.1650	0.2750	0.3467
μ	0.8200	0.6150	1.0250	-1.3602

7. Optimal control

As part of our intervention strategy, we are introducing three controls representing concrete measures to limit the spread of rumors and reinforce the impact of counter-rumors. The first control, rated u_1, corresponds to educating the population: it models efforts to develop critical thinking and information verification skills, which reduces the likelihood that an ignorant individual will become a spreader of the rumor. The second control variable, denoted u_2, represents awareness-raising actions and the dissemination of corrective information through the media, public institutions, or digital platforms. This intervention aims to limit the influence of rumors by promoting access to verified information and increasing the transition rate of exposed individuals to an informed or resistant state. And, the third control, u_3 reflects legal and enforcement measures, such as warnings, administrative sanctions, or legal proceedings against persistent disseminators of false information; it has the effect of directly reducing the capacity or duration of rumor transmission by active individuals. These three controls are assumed to take values in the interval $[0;1]$, where 0 corresponds to no intervention and 1 to maximum effort.

$$\begin{cases} I' &= \mu N - (1+u_2)\beta_1 \dfrac{IS}{N} - (1+u_1)\beta_2 \dfrac{IC}{N} - \mu I \\ A' &= (1-u_2)(1-\varepsilon_1)\beta_1 \dfrac{IS}{N} - \beta_3 \dfrac{AC}{N} - \gamma u_3 A - \mu A \\ S' &= \varepsilon_1(1-u_2)\beta_1 \dfrac{IS}{N} + \gamma u_3 A - \mu S \\ C' &= \xi E - \mu C \\ E' &= (1+u_1)\beta_2 \dfrac{IC}{N} + \beta_3 \dfrac{AC}{N} - (\xi+\mu)E \end{cases}$$

The objective is to minimize the objective functional

$$J(u_1,u_2,u_3) = \int_0^T \left[Q_1 S + Q_2 A + \frac{1}{2}(\alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2) \right] dt. \tag{7.1}$$

In other words, we seek the optimal control u_1^*, u_2^* *and* u_3^* such that

$$J(u_1^*,u_2^*,u_3^*) = \min_{(u_1,u_2,u_3)\in U_{ad}^T} J(u_1,u_2,u_3). \tag{7.2}$$

7.1. Existence of optimal control

Theorem 7.1 *There exists an optimal control $(u_1^*,u_2^*,u_3^*) \in U_{ad}^T$ for the system (7) such that*

$$J(u_1^*,u_2^*,u_3^*) = \min_{(u_1,u_2,u_3)\in U_{ad}^T} J(u_1,u_2,u_3).$$

7.1.1. Characterization of the optimal control. The necessary conditions for optimal control are obtained by applying Pontryagin's maximum principle to the Hamiltonian $H(t)$, defined at time t as:

$$H(t) = Q_1 S + Q_2 A + \frac{1}{2}(\alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2) + \lambda_1 \dot{I} + \lambda_2 \dot{A} + \lambda_3 \dot{S} + \lambda_4 \dot{C} + \lambda_5 \dot{E}. \tag{7.3}$$

Theorem 7.2 *Let (u_1^*,u_2^*,u_3^*) be an optimal control triple, and let $I^*(t), A^*(t), S^*(t), C^*(t), E^*(t)$ denote the corresponding optimal state trajectories of the system. Then, there exist adjoint functions $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t)$ such that the following conditions are satisfied:*

$$
\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H(t)}{\partial I(t)} = (1+u_2)\lambda_1\beta_1\frac{S}{N} - \lambda_2(1-u_2)\beta_1\frac{S}{N} + \varepsilon_1(1-u_2)\beta_1\frac{S}{N}(\lambda_2-\lambda_3) + (1+u_1)\beta_2\frac{C}{N}(\lambda_1-\lambda_5) + \lambda_1\mu,\\
\dot{\lambda_2}(t) &= -\frac{\partial H(t)}{\partial A(t)} = -Q_2 + \gamma u_3(\lambda_2-\lambda_3) + \beta_3\frac{C}{N}(\lambda_2-\lambda_5) + \lambda_2\mu,\\
\dot{\lambda_3}(t) &= -\frac{\partial H(t)}{\partial S(t)} = -Q_1 + (1-u_2)\beta_1\frac{I}{N}(\lambda_1-\lambda_2) + \varepsilon_1(1-u_2)\beta_1\frac{I}{N}(\lambda_2-\lambda_3) + \lambda_3\mu,\\
\dot{\lambda_4}(t) &= -\frac{\partial H(t)}{\partial C(t)} = (1+u_1)\beta_2\frac{I}{N}(\lambda_1-\lambda_5) + \beta_3\frac{A}{N}(\lambda_2-\lambda_5) + \lambda_4\mu,\\
\dot{\lambda_5}(t) &= -\frac{\partial H(t)}{\partial E(t)} = \xi(\lambda_5-\lambda_4) + \lambda_5\mu.
\end{aligned}
$$

Along with the transversality conditions specified at time T:

$$\lambda_1(T) = \lambda_4(T) = \lambda_5(T) = 0 \quad \textit{and} \quad \lambda_2(T) = Q_2\ ,\ \lambda_3(T) = Q_1.$$

Moreover, for all $t \in [0, T]$, the optimal control functions u_1^*, u_2^*, and u_3^* are expressed as follows:

$$
\begin{aligned}
u_1^* &= \max\{\min\{\frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5), 1\}, 0\}\\
u_2^* &= \max\{\min\{\frac{1}{\alpha_2}\beta_1\frac{IS}{N}\left[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)\right], 1\}, 0\}\\
u_3^* &= \max\{\min\{\frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)), 1\}, 0\}
\end{aligned}
$$

Proof: We use the Hamiltonian and Pontryagin's maximum principle to establish the adjoint equations and the conditions for transversality. Let $I(t) = I^*(t)$, $A(t) = A^*(t)$, $S(t) = S^*(t)$, $C(t) = C^*(t)$, $E(t) = E^*(t)$,

Using the optimality conditions, we conclude:

$$
\begin{aligned}
u_1 &= \frac{\partial H(t)}{\partial u_1(t)}\\
&= \alpha_1 u_1 - \lambda_1\beta_2\frac{IC}{N} + \lambda_5\beta_2\frac{IC}{N}.\\
u_2 &= \frac{\partial H(t)}{\partial u_2(t)}\\
&= \alpha_2 u_2 - \lambda_1\beta_1\frac{IS}{N} + \lambda_2\varepsilon_1\beta_1\frac{IS}{N} - \lambda_3\varepsilon_1\beta_1\frac{IS}{N} - \lambda_2\beta_1\frac{IS}{N}.\\
u_3 &= \frac{\partial H(t)}{\partial u_3(t)}\\
&= \alpha_3 u_3 - \lambda_2\gamma A + \lambda_3\gamma A.
\end{aligned}
\tag{7.4}
$$

Hence

$$
\begin{aligned}
\frac{\partial H}{u_1} = 0 \quad &\Rightarrow \quad u_1^* = \frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5),\\
\frac{\partial H}{u_2} = 0 \quad &\Rightarrow \quad u_2^* = \frac{1}{\alpha_2}\beta_1\frac{IS}{N}[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)].\\
\frac{\partial H}{u_3} = 0 \quad &\Rightarrow \quad u_3^* = \frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)),
\end{aligned}
$$

By exploiting the properties of the control space, we obtain the following results:

$$u_1^* = \begin{cases} 0 & \text{if } \frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5) \le 0 \\ \frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5) & \text{if } 0 < \frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5) < 1 \\ 1 & \text{if } \frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5) \ge 1 \end{cases}$$

$$u_2^* = \begin{cases} 0 & \text{if } \frac{1}{\alpha_2}\beta_1\frac{IS}{N}[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)] \le 0 \\ \frac{1}{\alpha_2}\beta_1\frac{IS}{N}[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)] & \text{if } 0 < \frac{1}{\alpha_2}\beta_1\frac{IS}{N}[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3+\lambda_2)] < 1 \\ 1 & \text{if } \frac{1}{\alpha_2}\beta_1\frac{IS}{N}[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)] \ge 1. \end{cases}$$

$$u_3^* = \begin{cases} 0 & \text{if } \frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)) \le 0 \\ \frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)) & \text{if } 0 < \frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)) < 1 \\ 1 & \text{if } \frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)) \ge 1. \end{cases}$$

We find that the optimal control is defined as follows based on the control space property:

$$\begin{aligned} u_1^* &= \max\{\min\{\frac{1}{\alpha_1}\beta_2\frac{IC}{N}(\lambda_1-\lambda_5),1\},0\} \\ u_2^* &= \max\{\min\{\frac{1}{\alpha_2}\beta_1\frac{IS}{N}\left[(\lambda_2+\lambda_1)+\varepsilon_1(\lambda_3-\lambda_2)\right],1\},0\} \\ u_3^* &= \max\{\min\{\frac{1}{\alpha_3}\gamma A(\lambda_2-\lambda_3)),1\},0\} \end{aligned}$$

This concludes the proof. □

8. Numerical simulation

This section presents the simulation results used to assess the influence on population dynamics of optimal control measures compared with an uncontrolled scenario. The fourth-order Runge–Kutta (RK4) method is used to solve numerically both models (2.1) and (7), with and without optimal control. The simulations use the parameter set listed in Table 2 and consider a time horizon of up to 20 hours. In the objective function, the weight constant and weight parameters are $Q_1 = 500$, $Q_2 = 800$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$ and $\alpha_3 = 0.05$. Figures 3, 5, 6 and 7 show the time series of the population state variables under both scenarios. The aim is to compare the effect of control on these variables in the model. The simulations indicate that optimal control increases the number of counter rumor individuals while simultaneously decreasing the number of acceptor and spreader individuals, whereas the educated class grows in the case where the controls exist and then declines when the control not existed. Clearly, the optimal control strategies eliminate spreading the rumors from the first time when the rumor circulates in Twitter, and the number of ignorant individuals remains stable. Based on these simulations, we conclude that control is highly effective at reducing the spread of rumor, if the control strategies are implemented, the spreading of the rumor may be terminated within about the first hour when the rumor circulate in social media.

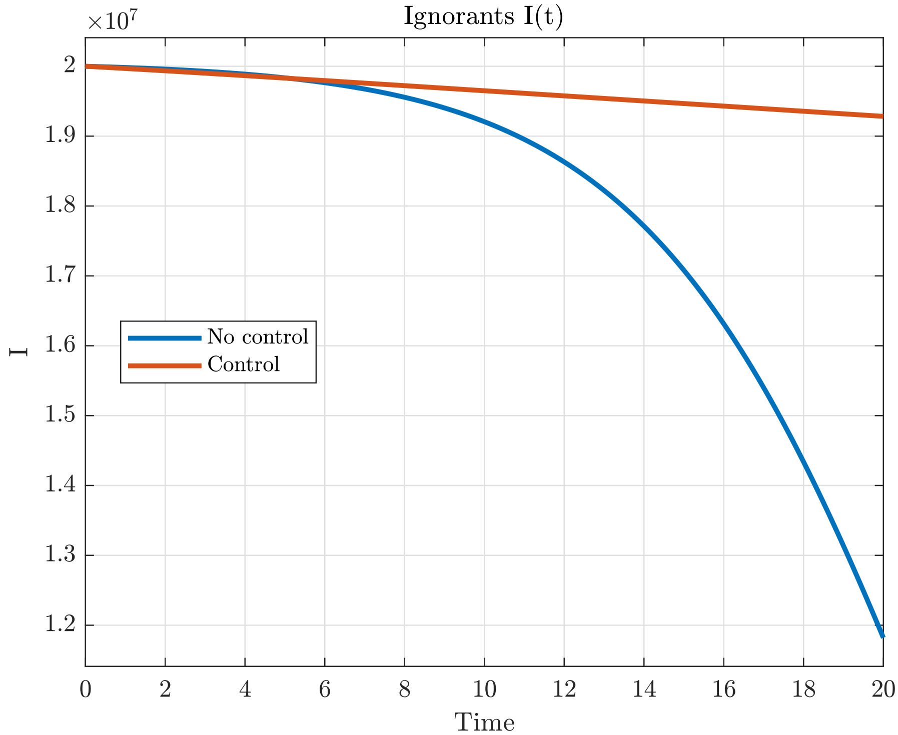

Figure 3: Comparison of the number of ignorant individuals in the controlled and uncontrolled cases.

9. Conclusion

In this article, we have developed and analyzed a deterministic mathematical model describing the concurrent propagation of a rumor and a counter-rumor, inspired by classical epidemiological models. Unlike traditional approaches that merely describe the spontaneous extinction of a rumor, our model explicitly introduces the counter-rumor as an active tool for regulating misinformation, making it an original contribution to the literature on rumor dynamics. Theoretical analysis has made it possible to determine the basic reproduction number $\Re_0$, characterize the equilibrium of the system, and rigorously demonstrate their overall stability using Lyapunov functions together with LaSalle's invariance principle.

In addition, the model has been enhanced by the integration of three realistic control strategies, namely critical thinking education, awareness campaigns, and legal measures. The corresponding control variables take values between 0 and 1, allowing the intensity of each intervention to be clearly represented and their effects to be directly compared.

Overall, this modeling framework illustrates how mathematical methods can serve as a reliable foundation for developing intervention strategies that balance effectiveness with economic considerations. It also opens the way for further extensions of the model, such as examining alternative scenarios, testing additional forms of intervention, or adapting the approach to other contexts of information dissemination.

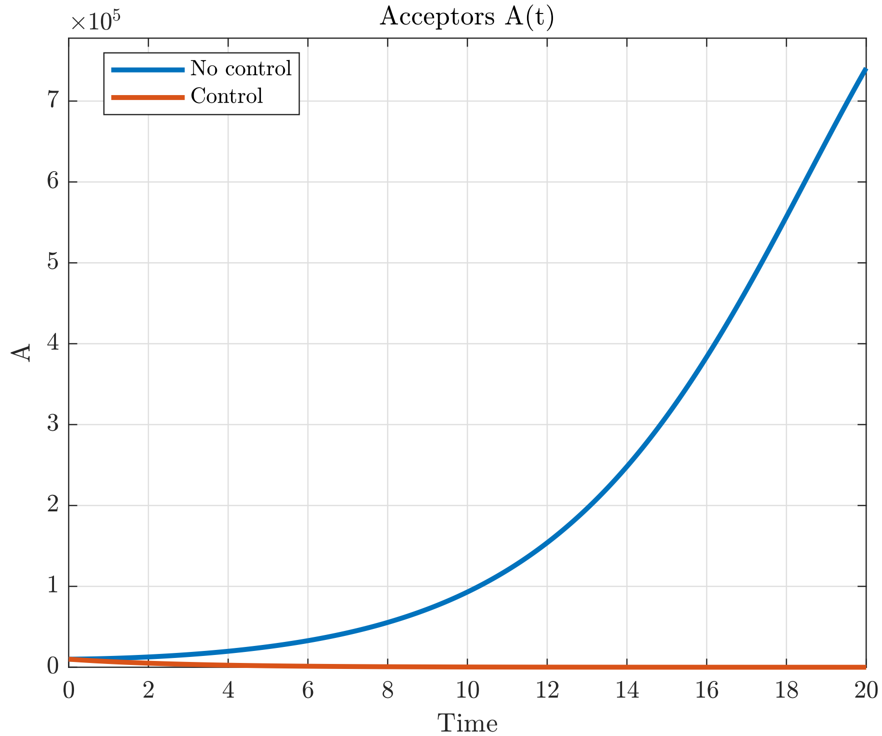

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Figure 4: Evolution of the acceptor population in the controlled and uncontrolled scenarios.

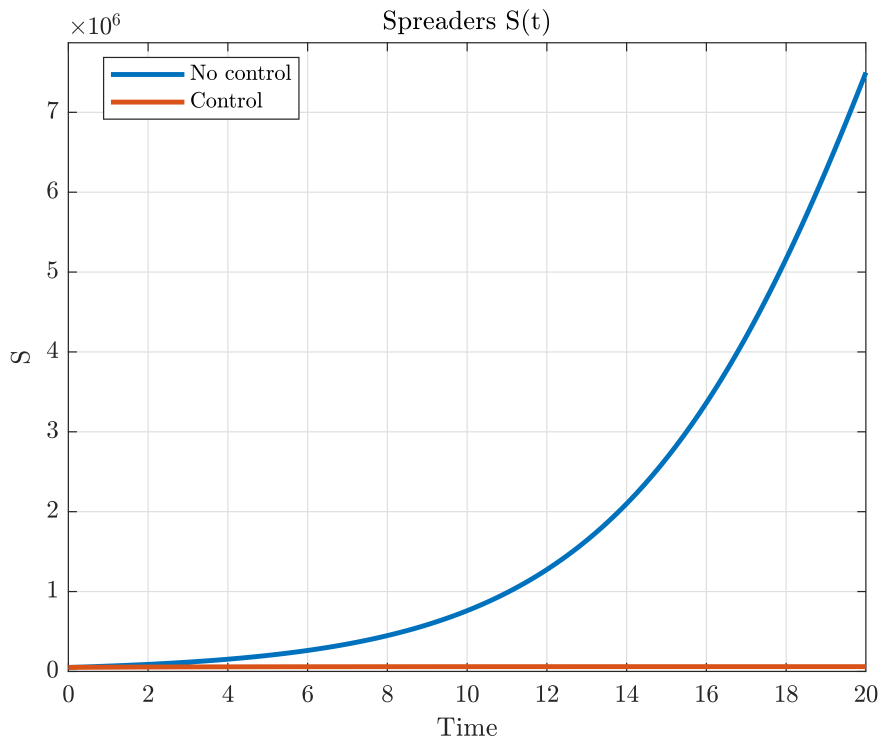

Figure 5: Evolution of Spreader individuals between controlled and uncontrolled scenario

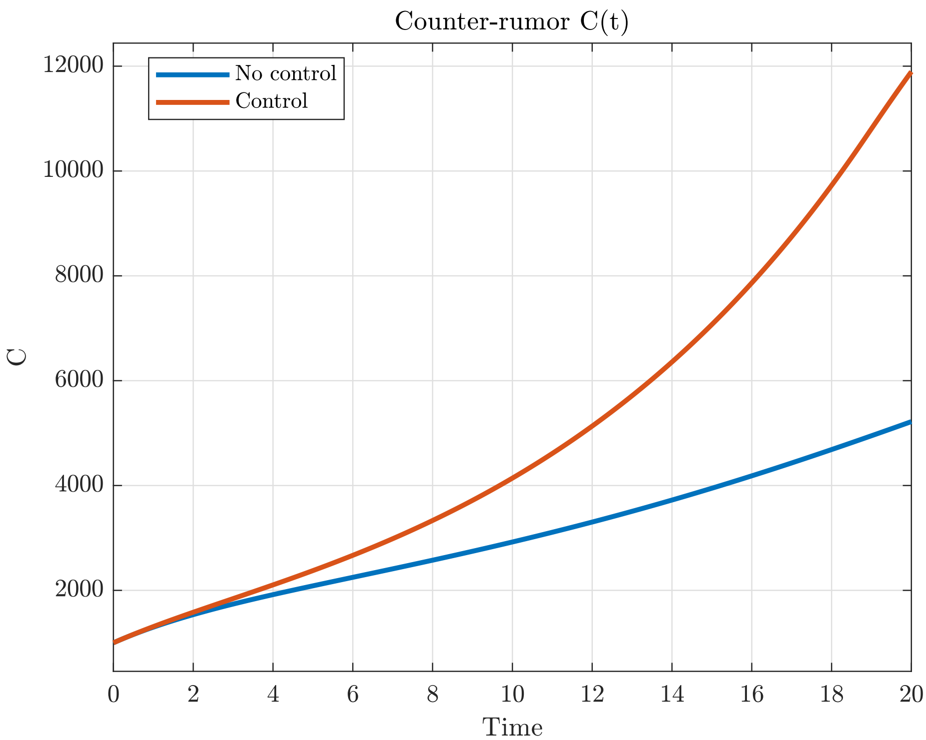

Figure 6: Evolution of Counter Rumor individuals between controlled and uncontrolled scenario

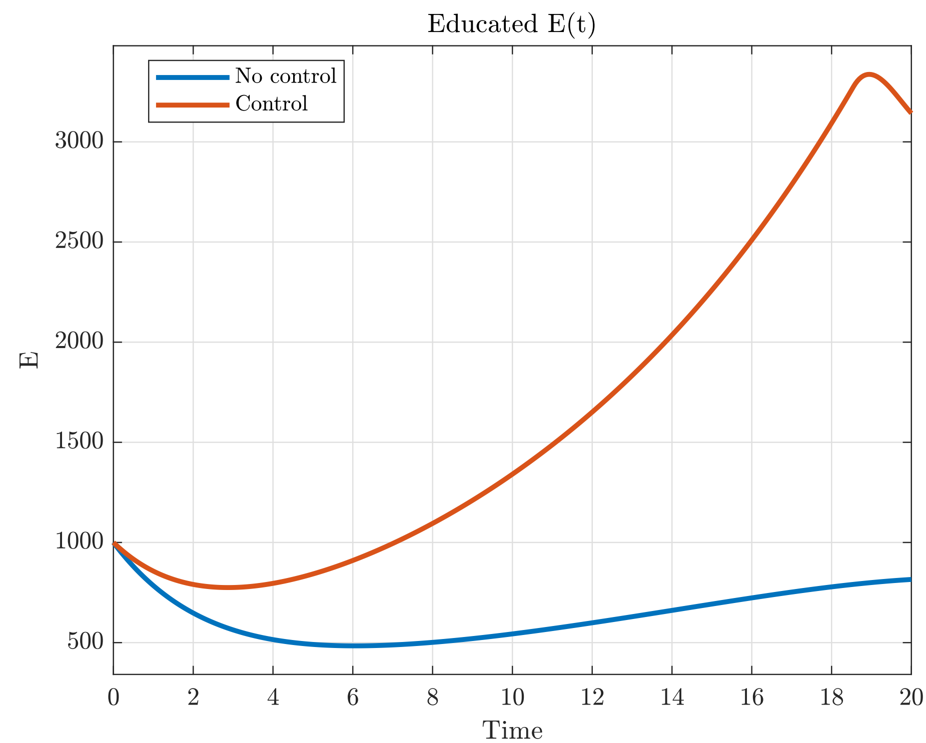

Figure 7: Evolution of educated individuals between controlled and uncontrolled scenario

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