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On the Extension of Abilov and Titchmarsh–Type Results to the Generalized Fourier–Bessel Transform

Mohamed Laamri*, Abdellatif Akhlidj and Souhir Arhilas

ABSTRACT: The main objective of this work is to provide estimates of abilov as well as a version of the titchmarsh theorem and to establish the equivalence between the K-functional and the regularity module associated with the generalized Fourier-Bessel transform in a class of functions in the space $\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$, where $1 < p \leq 2$. Here, $\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$ denotes the space of measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that $\mathcal{M}_n^{-1} f \in \mathrm{L}^p(\mathbb{R}_+, d\mu_\alpha(x))$, where $\mathcal{M}_n f(x) = x^{2n} f(x)$.

Keywords: Generalized Fourier-Bessel transform, generalized translation operators, modulus of continuity, K-functional.

Contents

1. Introduction

In [3], V. A. Abilov, F. V. Abilova and M. K. Kerimov showed via the Steklov's operator, two useful estimates for the Fourier transform in a class of functions in $\mathrm{L}^2(\mathbb{R}, dx)$.

In [4], the same authors, using the translation operators associated, with the classical Bessel operator, proved analogous estimates for the Fourier-Bessel transform in a class of functions in $\mathrm{L}^2(\mathbb{R}_+, d\mu_\alpha(x))$, where $d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx$.

In 2013 and 2014, the autors M. El Hamma, R. Daher and H. Lahlali extended estimates of the Bessel transform at a class of functions in the space $\mathrm{L}^p(\mathbb{R}_+, d\mu_\alpha(x))$, where $1 < p \leq 2$. In this work, we present a novel operational approach that allows us to extend several classical results in Bessel harmonic analysis. While these theorems have been already established in the classical setting, our method provides a more direct and systematic way to generalize them. By leveraging the operational framework, we are able to build on the well-known classical cases and extend the results to the generalized spaces $L^p_{\alpha,n}(\mathbb{R}_+)$, which will be defined and studied throughout this article. This approach not only simplifies the derivation of the generalized estimates but also highlights the underlying connections between the classical and the generalized settings.

In [3] and [4], V. A. Abilov, F. V. abilova and M. K. Kerimov have shown by the Steklov operator, two useful estimates of the Fourier transform of a class of functions belonging to the space $\mathrm{L}^2(\mathbb{R}, dx)$ characterized by the generalized modulus of continuity.

In this paper, we show similar results to those proved in ([1] and [8]), giving estimates of the generalized Fourier-Bessel transform of a class of functions belonging to the space $\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$, where $1 < p \leq 2$, $\alpha > \frac{-1}{2}$ and which are characterized by the generalized modulus of continuity.

Consider the singular differential operator of order two on the half line

$$\mathcal{B}_{\alpha,n} f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2} f(x),$$

where $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, \ldots$..

For $n = 0$, we obtain the classical Bessel operator denoted by $\mathcal{B}_\alpha$ giving by

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

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[*] Corresponding author.

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Let $\mathcal{M}_n$ the operator defined by:

$$\mathcal{M}_n f(x) = x^{2n} f(x).$$

by ([5], [6]) we have:

$$\mathcal{B}_{\alpha,n} = \mathcal{M}_n \circ \mathcal{B}_{\alpha+2n} \circ \mathcal{M}_n^{-1}.$$

Let $\mathrm{L}^p_\alpha(\mathbb{R}_+) = \mathrm{L}^p(\mathbb{R}_+, d\mu_\alpha(x))$ the space of measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\|f\|_{p,\alpha} = \left(\int_0^{+\infty} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty,$$

and
$\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$ the space of measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that $\mathcal{M}_n^{-1} f \in L^p_{\alpha+2n}(\mathbb{R}_+)$, i.e.

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}_n^{-1} f\|_{p,\alpha+2n} < \infty.$$

For $\alpha > -\frac{1}{2}$ and $n = 0, 1, 2, ...$, $\mathcal{M}_n$ is an isometric isomorphism from $\mathrm{L}^p_{\alpha+2n}(\mathbb{R}_+)$ onto $\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$ (see [6]).
For $\alpha > -\frac{1}{2}$, we introduce the normalized Bessel function of first kind and the order α defined by

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha}, \quad for \quad all \quad x \geq 0, \tag{1.1}$$

where $J_\alpha(x)$ is the Bessel function of the first kind and the order α and $\Gamma(x)$ is the gamma function (see [5]) defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad for \quad all \quad z \in \mathbb{C} \backslash (-\mathbb{N}). \tag{1.2}$$

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\alpha+1)} \left(\frac{z}{2} \right)^{2n} \quad for \quad all \quad z \in \mathbb{C}, \tag{1.3}$$

The function $y = j_\alpha$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0,$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$.
The function j_α is infinitely differentiable, even and moreover entire analytic.

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, we put

$$\varphi_{\alpha,n,\lambda}(x) = x^{2n} j_{\alpha+2n}(\lambda x).$$

By ([5], [6]) we have $\varphi_{\alpha,n,\lambda}$ satisfies the differential equation

$$\mathcal{B}_{\alpha,n}(\varphi_{\alpha,n,\lambda}) = -\lambda^2 \varphi_{\alpha,n,\lambda}.$$

The generalized Fourier-Bessel transform (see [5], [6]) is given by

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_0^{+\infty} f(x) \varphi_{\alpha,n,\lambda}(x) d\mu_{\alpha,n}(x), \quad for \quad all \quad f \in \mathrm{L}^1_{\alpha,n}(\mathbb{R}_+) \quad and \quad \lambda \geq 0,$$

where $d\mu_{\alpha,n}(x) = \frac{x^{2\alpha+1}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)} dx$.
We have the formula (see [5] and [6])

$$\mathcal{F}_{\alpha,n} = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}_n^{-1}$$

Let $f \in \mathrm{L}^1_{\alpha,n}(\mathbb{R}_+)$ and $\mathcal{F}_{\alpha,n}(f) \in L^1_{\alpha+2n}(\mathbb{R}_+)$ then, the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^{+\infty} \mathcal{F}_{\alpha,n}(f)(\lambda) \varphi_{\alpha,n,\lambda}(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = \frac{1}{2^{\alpha+2n}\Gamma(\alpha+2n+1)} \lambda^{2(\alpha+2n)+1} d\lambda.$$

Theorem 1.1 *([5], [6])*

1. *For every* $f \in \mathrm{L}^1_{\alpha,n}(\mathbb{R}_+) \cap \mathrm{L}^2_{\alpha,n}(\mathbb{R}_+)$ *we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 d\mu_{\alpha,n}(x) = \int_0^{+\infty} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

2. *The generalized Fourier-Bessel transform* $\mathcal{F}_{\alpha,n}$ *extends uniquely to an isometric isomorphism from* $\mathrm{L}^2_{\alpha,n}(\mathbb{R}_+)$ *onto* $\mathrm{L}^2_{\alpha+2n}(\mathbb{R}_+) = \mathrm{L}^2(\mathbb{R}_+, d\mu_{\alpha+2n}(x))$.

The generalized translation operators $\mathcal{T}_{\alpha,n,h}$, with $h > 0$ (see [5], [6]) is giving by

$$\mathcal{T}_{\alpha,n,h} = h^{2n} \mathcal{M}_n o \mathcal{T}_{\alpha+2n,h} o \mathcal{M}_n^{-1},$$

where $\mathcal{T}_{\alpha+2n,h}$ are the classical Bessel translation operators.

$$\mathcal{T}_{\alpha,h} f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2 + h^2 - 2xh cos\theta}) sin^{2\alpha}\theta d\theta,$$

where

$$c_\alpha = \left(\int_0^\pi sin^{2\alpha}\theta d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}$$

Let $f \in \mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$. Then for all $h \geq 0$, the function $\mathcal{T}_{\alpha,n,h} f \in \mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$, and

$$\|\mathcal{T}_{\alpha,n,h} f\|_{p,\alpha,n} \leq h^{2n} \|f\|_{p,\alpha,n}.$$

2. Main results

Proposition 2.1 *Let* $1 < p \leq 2$ *and* $g \in L^p_{\alpha,n}(\mathbb{R}_+)$ *then,* $\mathcal{M}_n \circ \mathcal{F}_{\alpha,n}(g) \in L^q_{\alpha,n}(\mathbb{R}_+)$ *and there exists a positive constant* $C > 0$ *such that*

$$\|\mathcal{M}_n \circ \mathcal{F}_{\alpha,n}(f)\|_{q,\alpha,n} \leq C \|f\|_{p,\alpha,n} \quad for \quad all \quad f \in L^p_{\alpha,n},$$

where $\frac{1}{p} + \frac{1}{q} = 1$

Proof: We have

$$g \in L^p_{\alpha,n}(\mathbb{R}_+) \Leftrightarrow \mathrm{M}_n^{-1} g \in L^p_{\alpha+2n}(\mathbb{R}_+)$$

Then, by the Hausdorff-Young inequality related to classic Bessel transform we have

$$\mathcal{F}_{\alpha+2n}(\mathcal{M}_n^{-1} g) \in L^q_{\alpha+2n}(\mathbb{R}_+), \quad where \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\mathcal{F}_{\alpha,n}(g) \in L^q_{\alpha+2n}(\mathbb{R}_+) \quad because \quad \mathcal{F}_{\alpha,n} = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}_n^{-1}.$$

Consequently

$$\mathcal{M}_n \circ \mathcal{F}_{\alpha,n}(g) \in L^q_{\alpha,n}(\mathbb{R}_+).$$

Moreover, by Riesz-Thorin theorem applying to operator $\mathcal{F}_{\alpha+2n}$, there is a constant $C > 0$ such that

$$\|\mathcal{F}_{\alpha+2n}(f)\|_{q,\alpha+2n} \leq C \|f\|_{p,\alpha+2n} \quad for \quad all \quad f \in L^p_{\alpha+2n}(\mathbb{R}_+).$$

Then,

$$\|\mathcal{F}_{\alpha+2n}(\mathcal{M}_n^{-1} g)\|_{q,\alpha+2n} \leq C \|\mathcal{M}_n^{-1} g\|_{p,\alpha+2n} \quad for \quad all \quad g \in L^p_{\alpha,n}(\mathbb{R}_+).$$

Hence,

$$\|\mathcal{F}_{\alpha,n}(g)\|_{q,\alpha+2n} \leq C \|g\|_{p,\alpha,n}.$$

The fact that the linear mapping $\mathcal{M}_n : L_{q,\alpha+2n}(\mathbb{R}_+) \to L^q_{\alpha,n}(\mathbb{R}_+)$ is an isometric isomorphism we conclude,

$$\|\mathcal{M}_n \circ \mathcal{F}_{\alpha,n}(g)\|_{q,\alpha,n} \leq C \|g\|_{p,\alpha,n} \quad for \quad all \quad g \in L^p_{\alpha,n}(\mathbb{R}_+).$$

□

The first and higher order finite difference of an function f are defined as follows:

Definition 2.1 ([7]) Let $k \in \mathbb{N}$. The k^{th} order modulus of continuity of a function $f \in L^p(\mathbb{R}_+, d\mu_\alpha(x))$ is defined as

$$\Omega_{k,p,\alpha}(f,\delta) = sup_{0<h\leq\delta}\|\Delta^k_{\alpha,h}f\|_{p,\alpha}, \tag{2.1}$$

where

$$\Delta^0_{\alpha,h}f = f$$

$$\Delta_{\alpha,h}f = (\mathcal{T}_{\alpha,h} - \mathcal{I})f$$

$$\Delta^k_{\alpha,h}f = (\mathcal{T}_{\alpha,h} - \mathcal{I})^k f$$

Where $\mathcal{I}$ is the identity in $L^p_\alpha(\mathbb{R}_+)$.

Definition 2.2 Let $k \in \mathbb{N}$. The k^{th} order generalized modulus of continuity of a function $f \in L^p_{\alpha,n}(\mathbb{R}_+)$ is defined as

$$\Omega_{k,p,\alpha,n}(f,\delta) = sup_{0<h\leq\delta}\|\Delta^k_{\alpha,n,h}f\|_{p,\alpha,n}, \tag{2.2}$$

where

$$\Delta^0_{\alpha,n,h}f = f, \tag{2.3}$$

$$\Delta_{\alpha,n,h}f = (\mathcal{T}_{\alpha,n,h} - h^{2n}\mathcal{I})f \tag{2.4}$$

$$\Delta^k_{\alpha,n,h}f = (\mathcal{T}_{\alpha,n,h} - h^{2n}\mathcal{I})^k f. \tag{2.5}$$

Where $\mathcal{I}$ is the identity in $L^p_{\alpha,n}(\mathbb{R}^+)$.

We denote in the classical Bessel harmonic analysis by
$\mathcal{W}^r_p(\mathcal{B}_\alpha)$, the class of functions f belongs to $\mathrm{L}^p_\alpha(\mathbb{R}_+)$ that have generalized derivatives in the sense of Levi (see [10])such that for all $j \in \{1, ..., r\}$ we have $\mathcal{B}^j_\alpha f \in \mathrm{L}^p_\alpha(\mathbb{R}_+)$.
And by
$\mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_\alpha)$, the class of functions f belongs to $\mathcal{W}^r_p(\mathcal{B}_\alpha)$ satisfying the estimate
$\Omega_{k,\alpha}((\mathcal{B}_\alpha)^r f, \delta) = \mathcal{O}\left(\psi(\delta^k\right)$ $as\ \delta \to 0$, where $r \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and ψ is a nonnegative function on $\mathbb{R}_+$.
For the classical Bessel operator $\mathcal{B}_\alpha$ we have $\mathcal{B}^0_\alpha f = f$, $\mathcal{B}^r_\alpha f = \mathcal{B}_\alpha(\mathcal{B}^{r-1}_\alpha f), r = 1, 2, ...$ We denote in the generalized Bessel harmonic analysis by
$\mathcal{W}^r_p(\mathcal{B}_{\alpha,n})$, the class of functions f belongs to $\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$ that have generalized derivatives in the sense of Levi (see [10]) such that for all $j \in \{1, ..., r\}$ we have $\mathcal{B}^j_{\alpha,n}f \in \mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$.
And by
$\mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_{\alpha,n})$, the class of functions f belongs to $\mathcal{W}^r_p(B_{\alpha,n})$ satisfying the estimate
$\Omega_{k,p,\alpha,n}((\mathcal{B}_{\alpha,n})^r f, \delta) = \mathcal{O}\left(\psi(\delta^k\right)$ $as\ \delta \to 0$, where $r \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and ψ is a nonnegative function on $\mathbb{R}_+$.
For the generalized Bessel operator $\mathcal{B}_{\alpha,n}$ we have $\mathcal{B}^0_{\alpha,n}f = f$, $\mathcal{B}^r_{\alpha,n}f = \mathcal{B}_{\alpha,n}\left(\mathcal{B}^{r-1}_{\alpha,n}f\right)$,
$r = 1, 2, ...$

Lemma 2.1 *Let* $f \in \mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)$ *we have*

i) $\Omega_{k,p,\alpha,n}(f,\delta) = \delta^{2nk}\Omega_{k,p,\alpha+2n}\left(\mathcal{M}^{-1}_n f, \delta\right)$.

ii) $\Omega_{k,p,\alpha,n}\left(\mathcal{B}^r_{\alpha,n}f,\delta\right) = \delta^{2nk}\Omega_{k,p,\alpha+2n}\left(\mathcal{B}^r_{\alpha+2n} \circ \mathcal{M}^{-1}_n f, \delta\right)$.

iii) $\mathcal{W}^r_p(\mathcal{B}_{\alpha,n}) = \mathcal{M}_n\left(\mathcal{W}^r_p(\mathcal{B}_{\alpha+2n})\right)$.

iv) $\mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_{\alpha,n}) = \mathcal{M}_n\left(\mathcal{W}^{r,k}_{p,\mathcal{M}^{-1}_n\psi}(\mathcal{B}_{\alpha+2n})\right)$

Proof:

i) We have

$$\begin{aligned}\Omega_{k,p,\alpha,n}(f,\delta) &= \sup_{0<h\leq\delta}\left\|\Delta^k_{h,\alpha,n}f\right\|_{p,\alpha,n}\\ \Delta^k_{h,\alpha,n} &= \left(\mathcal{T}_{h,\alpha,n}-h^{2n}E\right)^k\\ &= \left(h^{2n}\mathcal{M}_n\circ\mathcal{T}_{h,\alpha+2n}\circ\mathcal{M}_n^{-1}-h^{2n}E\right)^k\\ &= h^{2nk}\left(\mathcal{M}_n\circ\mathcal{T}_{h,\alpha+2n}\circ\mathcal{M}_n^{-1}-E\right)^k\\ &= h^{2nk}\mathcal{M}_n\circ\left(\mathcal{T}_{h,\alpha+2n}-E\right)^k\circ\mathcal{M}_n^{-1}\\ &= h^{2nk}\mathcal{M}_n\circ\Delta^k_{h,\alpha+2n}\circ\mathcal{M}_n^{-1}.\end{aligned}$$

Then, for $f\in \mathrm{L}^p_{\alpha,n}(\mathbb{R}^+)$ we have:

$$\begin{aligned}\left\|\Delta^k_{h,\alpha,n}f\right\|_{p,\alpha,n} &= h^{2nk}\left\|\mathcal{M}_n\circ\Delta^k_{h,\alpha+2n}\circ\mathcal{M}_n^{-1}f\right\|_{p,\alpha,n}\\ &= h^{2nk}\left\|\Delta^k_{h,\alpha+2n}\circ\mathcal{M}_n^{-1}f\right\|_{p,\alpha+2n}\end{aligned}$$

Hence,

$$sup_{0<h\leq\delta}\left\|\Delta^k_{h,\alpha,n}f\right\|_{p,\alpha,n}=\delta^{2nk}sup_{0<h\leq\delta}\left\|\Delta^k_{h,\alpha+2n}\mathcal{M}_n^{-1}f\right\|_{p,\alpha+2n}$$

Therefore:

$$\Omega_{k,p,\alpha,n}(f,\delta)=\delta^{2nk}\Omega_{k,p,\alpha+2n}\left(\mathcal{M}_n^{-1}f,\delta\right).$$

ii) according to i) we have:

$$\Omega_{k,p,\alpha,n}\left(\mathcal{B}^r_{\alpha,n}f,\delta\right)=\delta^{2nk}\Omega_{k,p,\alpha+2n}\left(\mathcal{M}_n^{-1}\circ\mathcal{B}^r_{\alpha,n}f,\delta\right)$$

Since

$$\mathcal{B}_{\alpha,n}=\mathcal{M}_n\circ\mathcal{B}_{\alpha+2n}\circ\mathcal{M}_n^{-1}.$$

Then we obtain

$$\mathcal{B}^r_{\alpha,n}=\mathcal{M}_n\circ\mathcal{B}^r_{\alpha+2n}\circ\mathcal{M}_n^{-1}$$

Consequently we deduce

$$\Omega_{k,p,\alpha,n}\left(\mathcal{B}^r_{\alpha,n}f,\delta\right)=\delta^{2nk}\Omega_{k,p,\alpha+2n}(\mathcal{B}^r_{\alpha+2n}\circ\mathcal{M}_n^{-1}f,\delta).$$

iv) We show that $f\in\mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_{\alpha,n})\Leftrightarrow f\in\mathcal{M}_n\left(\mathcal{W}^{r,k}_{p,\mathcal{M}_n^{-1}\psi}(\mathcal{B}_{\alpha+2n})\right)$.
$f\in\mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_{\alpha,n})\Leftrightarrow\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha+2n}(\mathbb{R}_+);\mathcal{M}_n\circ\mathcal{B}^j_{\alpha+2n}\circ\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha,n}(\mathbb{R}_+)\,\forall j\in\{1,...,r\}$ and $\Omega_{k,p,\alpha,n}(\mathcal{B}^r_{\alpha,n}f,\delta)=\mathcal{O}\left(\psi(\delta^k)\right)$ as $\delta\to0$
$\Leftrightarrow\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha+2n}(\mathbb{R}_+);\mathcal{B}^j_{\alpha+2n}\circ\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha+2n}(\mathbb{R}^+)\,\forall j\in\{1,...,r\}$
and $\delta^{2nk}\Omega_{k,p,\alpha+2n}\left(\mathcal{B}^r_{\alpha+2n}\circ\mathcal{M}_n^{-1}f,\delta\right)=\mathcal{O}\left(\psi(\delta^k)\right)$ as $\delta\to0$
$\Leftrightarrow\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha+2n}(\mathbb{R}_+);\mathcal{B}^j_{\alpha+2n}\circ\mathcal{M}_n^{-1}f\in\mathrm{L}^p_{\alpha+2n}(\mathbb{R}^+)\,\forall j\in\{1,...,r\}$
and $\Omega_{k,p,\alpha+2n}(\mathcal{B}^r_{\alpha+2n}\circ\mathcal{M}_n^{-1}f,\delta)=\mathcal{O}\left(M_n^{-1}(\psi)(\delta^k)\right)$ as $\delta\to0$
$\Leftrightarrow\mathcal{M}_n^{-1}f\in\mathcal{W}^{r,k}_{p,\mathcal{M}_n^{-1}\psi}(\mathcal{B}_{\alpha+2n})$
$\Leftrightarrow f\in\mathcal{M}_n\left(\mathcal{W}^{r,k}_{p,\mathcal{M}_n^{-1}\psi}(\mathcal{B}_{\alpha+2n})\right)$.

$\square$

On Estimates and Titchmarsh's Theorem

Theorem 2.1 *Let* $1 < p \leq 2$ *and* $f \in \mathrm{W}^{r,k}_{p,\psi}(\mathrm{B}_{\alpha,n})$ *then,*

$$\left(\int_N^{+\infty} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{q}} = O\left(N^{-2r}(\mathcal{M}_n^{-1}\psi)\left(\left(\frac{c}{N}\right)^k\right)\right) \text{ as } N \to +\infty,$$

where C *is a positive constant,* $r = 0, 1, 2...; k = 1, 2, ...;$ q *is the conjugate exponent of* p *and* ψ *is a positive function defined on* $[0, +\infty[$ *such that* $\psi(0) = 0$.

Proof: By lemma 2.2 we have:

$$f \in \mathcal{W}^{r,k}_{p,\psi}(\mathcal{B}_{\alpha,n}) \Leftrightarrow \mathcal{M}_n^{-1} f \in \mathcal{W}^{r,k}_{p,\mathcal{M}_n^{-1}\psi}(\mathcal{B}_{\alpha+2n})$$

Hence, by result related to classic Bessel harmonic analysis see [8] and by the Hausdorff-Young inequality we deduce that $\mathcal{F}_{\alpha+2n}(\mathcal{M}_n^{-1} f) \in L^q_{\alpha+2n}$ and

$$\left(\int_N^{+\infty} |\mathcal{F}_{\alpha+2n}(\mathcal{M}_n^{-1} f)(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{q}} = O\left(N^{-2r}(\mathcal{M}_n^{-1}\psi)\left(\left(\frac{c}{N}\right)^k\right)\right)$$

The fact that $\mathcal{F}_{\alpha,n} = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}_n^{-1}$ we obtain the result. □

Corollary 2.1 *For* $\psi(t) = t^\nu$, *with* $\nu > 0$ *and* $f \in \mathcal{W}^{r,k}_{p,t^\nu}(\mathcal{B}_{\alpha,n})$ *we have:*

$$\left(\int_N^{+\infty} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{q}} = \mathcal{O}\left(N^{-2r+2nk-k\nu}\right) \text{ as } N \to +\infty,$$

where $r = 0, 1, 2...; k = 1, 2, ...;$.

Theorem 2.2 *For* $\psi(t) = t^\nu$. *The follows conditions are equivalents:*

i) $\left(\int_N^{+\infty} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2}} = O(N^{2kn-2r-k\nu})$ *as* $N \to +\infty$.

ii) $f \in \mathcal{W}^{r,k}_{2,t^\nu}(\mathcal{B}_{\alpha,n})$.

Proof: For $\psi(t) = t^\nu$ we have $\mathcal{M}_n^{-1}\psi(t) = t^{-2n+\nu}$ and

$$f \in \mathcal{W}^{r,k}_{2,t^\nu}(\mathcal{B}_{\alpha,n}) \Leftrightarrow \mathcal{M}_n^{-1} f \in \mathcal{W}^{r,k}_{2,t^{-2n+\nu}}(\mathcal{B}_{\alpha+2n}).$$

By classic result see [1] we have:

$$\mathcal{M}_n^{-1} f \in \mathcal{W}^{r,k}_{2,t^{-2n+\nu}}(\mathcal{B}_{\alpha+2n}) \Leftrightarrow \left(\int_N^{+\infty} |\mathcal{F}_{\alpha+2n}(\mathcal{M}_n^{-1} f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2}} = O(N^{2kn-2r-k\nu}).$$

The fact that $\mathcal{F}_{\alpha,n} = \mathcal{F}_{\alpha+2n} \circ \mathcal{M}_n^{-1}$ we deduce

$$f \in \mathcal{W}^{r,k}_{2,t^\nu}(\mathcal{B}_{\alpha,n}) \Leftrightarrow \left(\int_N^{+\infty} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{2}} = O(N^{2kn-2r-k\nu}).$$

□

Let $\mathcal{W}^r_p(B_{\alpha,n})$ the space of functions $f \in \mathrm{L}^p_{\alpha,n}$ such that $\mathcal{B}^j_{\alpha,n} f \in \mathcal{L}^p_{\alpha,n}$ for all $j \in \{0, 1, ..., r\}$.

Theorem 2.3 *Let $f \in \mathcal{W}_p^r(B_{\alpha,n})$, where $1 < p \leq 2$ such that*

$$\left\|\Delta_{h,\alpha,n}^k \mathcal{B}_{\alpha,n}^r f\right\|_{p,\alpha,n} = \mathcal{O}(h^{\gamma+2nk}) \quad as \quad h \to 0 \quad where \quad 0 < \gamma \leq 2k$$

Then, $\mathcal{M}_n \circ \mathcal{F}_{\alpha,n}(f) \in L_{\alpha,n}^{\beta}$, where

$$\frac{2p(\alpha+2n)+2p}{2p+2(\alpha+2n)(p-1)+2rp+\gamma p-2} < \beta \leq \frac{p}{p-1}.$$

Proof: We have
$f \in \mathcal{W}_p^r(\mathcal{B}_{\alpha,n}) \Leftrightarrow \mathcal{M}_n^{-1} f \in$
$mathcalW_p^r(\mathcal{B}_{\alpha+2n})$ and

$$\left\|\Delta_{h,\alpha,n}^k \mathcal{B}_{\alpha,n}^r f\right\|_{p,\alpha,n} = h^{2nk} \left\|\Delta_{h,\alpha+2n}^k \mathcal{B}_{\alpha+2n}^r (\mathcal{M}_n^{-1} f)\right\|_{p,\alpha+2n}$$

Hence, by hypothesis we deduce that
$\left\|\Delta_{h,\alpha+2n}^k \mathcal{B}_{\alpha+2n}^r (\mathcal{M}_n^{-1} f)\right\|_{p,\alpha+2n} = \mathcal{O}(h^{\gamma})$ as $h \to 0$. By [9] $\mathcal{F}_{\alpha+2n} \circ \mathcal{M}_n^{-1} f \in L_{\alpha+2n}^{\beta}$, where

$$\frac{2p(\alpha+2n)+2p}{2p+2(\alpha+2n)(p-1)+2rp+\gamma p-2} < \beta \leq \frac{p}{p-1}.$$

□

Equivalence of K-functional and moudulus of smoothness associated with the generalized Fourier–Bessel operator

Definition 2.3 Let $1 \leq p \leq \infty$, $r \in \mathbb{N}$ and $\alpha \geq 0$. The K-functional associated, with the generalized Bessel differential operator $B_{\alpha,n}$ is defined by

$$\mathcal{K}_r^{(\alpha,n)}(f,t;L_{\alpha,n}^p(\mathbb{R}_+);\mathcal{W}_{\alpha,n,p}^r)_p = \inf_{g \in \mathcal{W}^r(\mathcal{B}_{\alpha,n})} \left(\|f-g\|_{L_{\alpha,n}^p(\mathbb{R}_+)} + t^r \left\|\mathcal{B}_{\alpha,n}^r g\right\|_{L_{\alpha,n}^p(\mathbb{R}_+)}\right), t > 0,$$

where $\mathcal{W}^r(\mathcal{B}_{\alpha,n})$ denotes the Sobolev-type space associated, with the generalized Bessel operator $B_{\alpha,n}$, defined by

$$\mathcal{W}^r(\mathcal{B}_{\alpha,n}) = \left\{f \in L_{\alpha,n}^p(\mathbb{R}_+) : \mathcal{B}_{\alpha,n}^k f \in L_{\alpha,n}^p(\mathbb{R}_+),\ k = 1,\ldots,r\right\}.$$

we denote $\mathcal{K}_r^{(\alpha,n)}(f,t)_p = \mathcal{K}_r^{(\alpha,n)}(f,t;L_{\alpha,n}^p(\mathbb{R}_+);\mathcal{W}_{\alpha,n,p}^r)_p$ and $\mathcal{W}_{\alpha,n,p}^r = \mathcal{W}^r(\mathcal{B}_{\alpha,n})$

Proposition 2.2 *Let $1 \leq p \leq \infty$, $r \in \mathbb{N}$ and $\alpha \geq 0$. Let M_n be the intertwining operator between the generalized Bessel operator $\mathcal{B}_{\alpha,n}$ and the classical Bessel operator $\mathcal{B}_\alpha$. Then, for all $f \in L_{\alpha,n}^p(\mathbb{R}_+)$ and all $t > 0$, we have*

$$\mathcal{K}_r^{(\alpha,n)}(f,t)_p = \mathcal{K}_r^{(\alpha+2n)}\left(M_n^{-1} f, t\right)_p,$$

where $\mathcal{K}_r^{(\alpha,n)}$ denotes the K-functional associated, with $B_{\alpha,n}$ and $\mathcal{K}_r^{(\alpha)}$ is the classical Bessel K-functional associated, with B_α.

Proof:

Recall that the $\mathcal{K}$-functional associated, with the generalized Bessel operator $\mathcal{B}_{\alpha,n}$ is defined by

$$\mathcal{K}_r^{(\alpha,n)}(f,t)_p = \inf_{g \in \mathcal{W}_{\alpha,n,p}^r} \left(\|f-g\|_{L_{\alpha,n}^p} + t^r \|\mathcal{B}_{\alpha,n}^r g\|_{L_{\alpha,n}^p}\right).$$

Let $g \in \mathcal{W}_{\alpha,n,p}^r$. Since $\mathcal{B}_{\alpha,n} = \mathcal{M}_n \mathcal{B}_\alpha \mathcal{M}_n^{-1}$, we have

$$\mathcal{B}_{\alpha,n}^r g = \mathcal{M}_n \mathcal{B}_\alpha^r \mathcal{M}_n^{-1} g.$$

we obtain

$$\|f-g\|_{L^p_{\alpha,n}} = \|\mathcal{M}_n^{-1}(f) - \mathcal{M}_n^{-1}(g)\|_{L^p_{\alpha+2nn}}$$

and

$$\|\mathcal{B}^r_{\alpha,n} g\|_{L^p_{\alpha,n}} = \|\mathcal{M}_n^{-1} B^r_{\alpha,n} g\|_{L^p_{\alpha+2n}} = \|\mathcal{B}^r_{\alpha+2n}\mathcal{M}_n^{-1}(g)\|_{L^p_{\alpha+2n}}$$

Therefore,

$$\mathcal{K}_r^{(\alpha,n)}(f,t)_p = \inf_{\mathcal{M}_n^{-1} g \in \mathcal{W}^r_{\alpha,p}} \left(\|\mathcal{M}_n^{-1} f - \mathcal{M}_n^{-1} g\|_{L^p_{\alpha+2n}} + t^r \|\mathcal{B}^r_{\alpha+2n}\mathcal{M}_n^{-1} g\|_{L^p_{\alpha+2n}} \right),$$

where $\mathcal{W}^r_{\alpha,p} = \mathcal{W}^r_p(\mathcal{B}_\alpha)$. So :

$$\mathcal{K}_r^{(\alpha,n)}(f,t)_p = \mathcal{K}_r^{(\alpha)}\left(M_n^{-1} f, t\right)_p.$$

This completes the proof. □

Proposition 2.3 (Extension generalized case) *Let $f \in L^p_{\alpha,n}(\mathbb{R}_+)$, and denote by*

$$\Omega_{r,p,\alpha,n}(f,t) := \sup_{|h|\le t} \|\Delta^r_{h,\alpha,n} f\|_{L^p_{\alpha,n}}$$

the generalized Bessel modulus of continuity. Then there exist constants $c_1, c_2 > 0$ such that, for all $t > 0$,

$$c_1 t^{-2nm}\, \Omega_{m,2,\alpha,n}(f,t) \le \mathcal{K}_m^{(\alpha,n)}(f,t^{2m})_2 \le c_2 t^{-2nm}\, \Omega_{m,2,\alpha,n}(f,t),$$

where $f \in L^2_{\alpha,n}(\mathbb{R}_+)$

Proof: Let $f \in L^2_{\alpha,n}(\mathbb{R}_+)$. By the intertwining properties of the operator M_n, the generalized $K-$functional satisfies

$$\mathcal{K}_m^{(\alpha,n)}(f,t)_2 = \mathcal{K}_m^{(\alpha+2n)}\left(M_n^{-1} f, t\right)_2, \tag{9}$$

In the classical case (that is, for $n = 0$), it is well known that the $K-$functional is equivalent to the modulus of smoothness, namely, there exist constants $c_1, c_2 > 0$ such that (by [11])

$$c_1\ \Omega_{m,2,\alpha+2n}(g,t) \le \mathcal{K}_m^{(\alpha+2n)}(g,t^{2m})_2 \le c_2\ \Omega_{m,2,\alpha+2n}(g,t), \tag{10}$$

for all $g \in L^2_{\alpha+2n}(\mathbb{R}_+)$ and all $t > 0$.

Applying, with $g = \mathcal{M}_n^{-1} f$ and using the relation between the generalized and classical moduli of smoothness,

$$\Omega_{m,2,\alpha+2n}\left(\mathcal{M}_n^{-1} f, t\right) = t^{-2mn}\, \Omega_{m,2,\alpha,n}(f,t), \tag{11}$$

we obtain

$$c_1\, t^{-2mn}\, \Omega_{m,2,\alpha,n}(f,t) \le \mathcal{K}_m^{(\alpha,n)}(f,t^{2m})_2 \le c_2\, t^{-2mn}\, \Omega_{m,2,\alpha,n}(f,t).$$

This completes the proof. □

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Mohamed Laamri,
Department of Mathematics and informatics, Faculty of Sciences Aïn Chock,
Hassan II University of Casablanca, B.P 5366 Mâarif,
Morocco.
E-mail address: `laamri.mohamed68@gmail.com`

and

Abdellatif Akhlidj,
Department of Mathematics and informatics, Faculty of Sciences Aïn Chock,
Hassan II University of Casablanca, B.P 5366 Mâarif,
Morocco.
E-mail address: `akhlidjabdellatif2@gmail.com`

and

Souhir Arhilas,
Department of Mathematics and informatics, Faculty of Sciences Aïn Chock,
Hassan II University of Casablanca, B.P 5366 Mâarif,
Morocco.
E-mail address: `arhilassouhir@gmail.com`