Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 18** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.82952

Translation Operator and Heat Equation Analysis for the Generalized Linear Canonical Fourier-Bessel Transform

Zakaria Sadik, Abdellatif Akhlidj and Athmane Boumazourh

ABSTRACT: In this work, we introduce a new translation operator naturally associated with the generalized linear canonical Fourier–Bessel transform $\mathcal{F}^m_{\alpha,n}$. This operator is constructed through an appropriate Cauchy problem involving a generalized Bessel-type differential operator and extends several known translation structures in harmonic analysis. We establish its main analytical properties and use it to define a convolution structure adapted to the generalized linear canonical Fourier–Bessel framework. Furthermore, we apply this convolution approach to the study of the heat equation governed by the conjugate of the generalized Bessel-type operator $\Delta^m_{\alpha,n}$. An explicit representation of the solution is obtained via a generalized heat kernel, highlighting the effectiveness of the proposed method and its potential applications.

Keywords: Generalized Canonical Fourier–Bessel transform, generalized translation operator, convolution product, heat equation.

Contents

1. Introduction

Integral transforms play a central role in harmonic analysis and its applications, particularly in mathematical physics, signal processing, and quantum mechanics [2,11]. Among these, the generalized linear canonical Fourier–Bessel transform (GLCFBT) extends classical transforms such as the Fourier and Hankel transforms by incorporating matrix parameters $m \in SL(2, \mathbb{R})$ [1,10], providing greater analytical flexibility. In this context, various studies have investigated generalizations and properties related to translation operators and associated differential equations.

Several contributions have focused on harmonic analysis in the framework of the generalized linear canonical Fourier–Bessel transform. In particular, [1] established a comprehensive analytical framework, including inversion formulas, Plancherel-type results, and several uncertainty principles. Nevertheless, despite these advances, the study of translation operators and convolution structures in this generalized setting remains limited.

Motivated by these observations, the present paper extends the theory by introducing a new translation operator associated with the (GLCFBT). This operator is coupled with a convolution structure

2020 *Mathematics Subject Classification:* 33D15, 42A38, 47A05.
Submitted March 24, 2026. Published June 22, 2026.

that generalizes classical constructions and enables the treatment of open problems related to the heat equation governed by the conjugate generalized Bessel-type operator $\overline{\Delta^m_{\alpha,n}}$. Specifically, for a function $f \in \mathcal{E}^2([0,+\infty[)$, we define the generalized translation operator $T^m_{x,n} f(y)$, for $(x,y) \geq 0$, $m \in SL(2,\mathbb{R})$, and $n \in \mathbb{N}$, as the solution of the Cauchy problem:

$$\begin{cases} \Delta^{\alpha,m}_{x,n} u(x,y) = \Delta^{\alpha,m}_{y,n} u(x,y), \\ u(x,0) = f(x), \quad \dfrac{\partial}{\partial x} u(x,0) = 0, \end{cases}$$

where $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ $(ad - bc = 1)$ with $b \neq 0$, and

$$\Delta^m_{\alpha,n} = \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b} \right) - \frac{4n(\alpha+n)}{x^2},$$

with $\Delta^{\alpha,m}_{x,n}$ acting on the variable x.

We show that the unique solution of this Cauchy problem admits an explicit representation in terms of the classical translation operator associated with the Bessel-type operator $\Delta^m_{\alpha+2n}$:

$$T^{\alpha,m}_{x,n} f(y) = (xy)^{2n}\, T^m_{\alpha+2n,x}(\mathcal{M}^{-1} f)(y). \tag{1.1}$$

Based on this operator, we define a generalized convolution product $\underset{\alpha,m,n}{*}$ associated with $\Delta^m_{\alpha,n}$:

$$f \underset{\alpha,m,n}{*} g = \int_0^{+\infty} \left[T^{\alpha,m}_{x,n} f(y) \right] \left[e^{-i\frac{a}{b}y^2} g(y) \right] y^{2\alpha+1} dy, \tag{1.2}$$

and investigate its main algebraic and analytical properties. These results are then applied to the study of the heat equation associated with $\overline{\Delta^m_{\alpha,n}}$:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \sigma \overline{\Delta^m_{\alpha,n}}\, u(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = f(x). \end{cases}$$

The paper is organized as follows. Section 2 recalls the necessary background on the linear canonical Fourier–Bessel transform and associated function spaces. Section 3 introduces the generalized translation operator, while Section 4 presents the corresponding convolution product. Section 5 addresses the heat equation related to the generalized Bessel-type operator.

2. Preliminaries

In this section, we briefly recall some basic notions related to the linear and generalized canonical Fourier–Bessel transforms, and describe the translation and convolution structures associated with the differential operator Δ^m_α. More details can be found in [1,5,8]. Throughout the paper, we assume that $\alpha > -1/2$.

2.1. Functional framework

We denote by $\mathscr{C}(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ and by $\mathscr{C}_*(\mathbb{R})$ the space of even smooth functions.

For $1 \leq p \leq \infty$, the weighted Lebesgue space $\mathcal{L}_{p,\alpha}$ consists of measurable functions f on $[0,\infty[$ such that

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(y)|^p\, y^{2\alpha+1}\, dy \right)^{1/p} < \infty.$$

In particular, $\mathcal{L}_{2,\alpha}$ is a Hilbert space with inner product

$$\langle f, g \rangle_\alpha = \int_0^\infty f(y)\overline{g(y)}\, y^{2\alpha+1}\, dy.$$

We also denote by $\mathcal{S}_*(\mathbb{R})$ the even Schwartz space.

2.2. Linear canonical Fourier–Bessel transform

Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ with $b \neq 0$. For $f \in \mathcal{L}_{1,\alpha}$, the linear canonical Fourier–Bessel transform is defined by

$$\mathcal{F}_\alpha^m f(y) = \frac{c_\alpha}{(ib)^{\alpha+1}} \int_0^\infty K_\alpha^m(y,x) f(x) x^{2\alpha+1} dx, \tag{2.1}$$

where $c_\alpha^{-1} = 2^\alpha \Gamma(\alpha+1)$ and

$$K_\alpha^m(y,x) = e^{\frac{i}{2}(\frac{a}{b}x^2 + \frac{d}{b}y^2)} j_\alpha(\frac{xy}{b}),$$

Here j_α denotes the normalized Bessel function (see [5,8]).

2.3. Generalized linear canonical Fourier–Bessel transform

Let $n \geq 0$. Consider the multiplication operator

$$\mathcal{M} f(x) = x^{2n} f(x),$$

which induces an isometric isomorphism between the weighted spaces $\mathcal{L}_{p,\alpha+2n}$ and $\mathcal{L}^p_{\alpha,n}$, the latter being endowed with the norm

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}^{-1} f\|_{p,\alpha+2n}. \tag{2.2}$$

For $f \in \mathcal{L}^1_{\alpha,n}$, the generalized canonical Fourier–Bessel transform is given by

$$\mathcal{F}^m_{\alpha,n} f(y) = \frac{c_{\alpha+2n}}{(ib)^{\alpha+2n+1}} \int_0^\infty K^m_{\alpha,n}(y,x) f(x) x^{2\alpha+1}\, dx. \tag{2.3}$$

where $K^m_{\alpha,n}(y,x) = x^{2n} K^m_{\alpha+2n}(y,x)$.
The associated differential operator is

$$\Delta^m_{\alpha,n} = \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x\right)\frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b}\right) - \frac{4n(\alpha+n)}{x^2}.$$

The operator $\Delta^m_{\alpha,n}$ is symmetric on $\mathcal{S}*(\mathbb{R})$ and is related to $\Delta^m_{\alpha+2n}$ through $\mathcal{M}$ (see [1]).

2.4. Translation operator

The translation operator associated with the linear canonical Fourier–Bessel transform was introduced in [5,8]. For suitable functions f, it is defined by

$$T_x^{\alpha,m} f(y) = e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} T_x^\alpha \left[s \mapsto e^{-\frac{i}{2}\frac{a}{b}s^2} f(s) \right](y), \tag{2.4}$$

where T_x^α denotes the classical Bessel translation operator.

2.5. Convolution product

Using this operator, one defines the convolution product by

$$f \underset{\alpha,m}{*} g(x) = \frac{(ib)^{\alpha+1}}{c_\alpha} \int_0^\infty \big(T_x^{\alpha,m} f(y)\big)\big(e^{-i\frac{a}{b}y^2} g(y)\big) y^{2\alpha+1}\, dy, \tag{2.5}$$

whenever the integral is well defined.

3. Generalized translation operator associated with $\Delta_{\alpha,n}^m$

Let

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \qquad b \neq 0,$$

and let $\alpha > \frac{-1}{2}$ and $n \in \mathbb{N}$. The generalized translation related to the operator $\Delta_{\alpha,n}^m$ is introduced via a symmetric evolution equation.

For a given function $f \in \mathscr{C}(\mathbb{R})$, consider the following Cauchy problem:

$$\begin{cases} \Delta_{x,n}^{m,\alpha} u(x,y) = \Delta_{y,n}^{m,\alpha} u(x,y), \\ u(x,0) = f(x), \\ \partial_x u(x,0) = 0. \end{cases} \tag{3.1}$$

This problem admits a unique solution, which allows us to define the generalized translation operator $T_{x,n}^{\alpha,m}$ by

$$T_{x,n}^{\alpha,m} f(y) = (xy)^{2n}\, T_x^{\alpha+2n,m}\big(\mathcal{M}^{-1} f\big)(y), \tag{3.2}$$

where $T_x^{\alpha+2n,m}$ denotes the classical translation associated with $\Delta_{\alpha+2n}^m$ and $\mathcal{M}$ is the intertwining operator introduced previously.

Definition 3.1 *Let $m \in SL(2,\mathbb{R})$ such that $b \neq 0$. For each $f \in \mathscr{C}(\mathbb{R})$, we define the generalized translation operators associated with the operator $\Delta_{\alpha,n}^m$ by :*

$$T_{x,n}^{\alpha,m} f = (xy)^{2n} T_x^{\alpha+2n,m}(\mathcal{M}^{-1} f)(y). \tag{3.3}$$

Remark 3.1 It is easy to see that

$$T_{x,n}^{\alpha,m} f = x^{2n}\left[\mathcal{M} \circ T_x^{\alpha+2n,m} \circ \mathcal{M}^{-1}\right] f.$$

Proposition 3.1 *The operator $T_{x,n}^{\alpha,m}$ satisfies the following properties:*

(1) ***Normalization and symmetry.***

$$T_{0,n}^{\alpha,m} f = f, \qquad T_{x,n}^{\alpha,m} f(y) = T_{y,n}^{\alpha,m} f(x).$$

(2) ***Linearity.*** *For all $\alpha, \beta \in \mathbb{C}$,*

$$T_{x,n}^{\alpha,m}(\alpha f + \beta g) = \alpha T_{x,n}^{\alpha,m} f + \beta T_{x,n}^{\alpha,m} g.$$

(3) ***Support preservation.*** *If $\mathrm{supp}(f) \subset [0,r]$, then $T_{x,n}^{\alpha,m} f(y) = 0$ whenever $|x-y| \geq r$.*

(4) ***Kernel product formula.*** *For all $x, y, z \geq 0$,*

$$T_{x,n}^{\alpha,m}\left[K_{\alpha,n}^m(y,\cdot)\right](z) = e^{-\frac{i}{2}\frac{d}{b}y^2} K_{\alpha,n}^m(y,x)\, K_{\alpha,n}^m(y,z).$$

(5) ***Integral representation.*** *For $x, y > 0$,*

$$T_{x,n}^{\alpha,m} f(y) = \int_0^\infty e^{-i\frac{a}{b}z^2} f(z) W_{\alpha,n}^m(x,y,z)\, z^{2\alpha+1}\, dz,$$

where $W_{\alpha,n}^m(x,y,z) = (xyz)^{2n} W_{\alpha+2n}^m(x,y,z)$.

6) ***Continuity of translation operator $T_{x,n}^{\alpha,m}$:*** *The operator $T_{x,n}^{\alpha,m}$ is continuous from :*

- *$\mathscr{C}(\mathbb{R})$ into itself.*
- *$\mathscr{C}_*(\mathbb{R})$ into itself.*

- $\mathcal{L}^p_{\alpha,n}$ *into itself. More precisely, for all* $f \in \mathcal{L}^p_{\alpha,n}$*,* $p \in [1,+\infty]$*, and* $x \geqslant 0$*, the function* $T^{\alpha,m}_{x,n} f$ *is defined almost everywhere on* $[0,+\infty]$*, belongs to* $\mathcal{L}^p_{\alpha,n}$ *and we have :*

$$\|T^{\alpha,m}_{x,n} f\|_{p,\alpha,n} \leqslant x^{2n}\|f\|_{p,\alpha,n}. \tag{3.4}$$

7) ***Commutativity :***

- $T^{\alpha,m}_{x,n}$ *commutes with itself in* $\mathscr{C}(\mathbb{R})$ *:*

$$T^{\alpha,m}_{x,n} \circ T^{\alpha,m}_{y,n} = T^{\alpha,m}_{y,n} \circ T^{\alpha,m}_{x,n}.$$

- *It also commutes with* $\Delta^m_{\alpha,n}$ *in* $\mathscr{C}_*(\mathbb{R})$ *:*

$$\Delta^m_{\alpha,n} \circ T^{\alpha,m}_{x,n} = T^{\alpha,m}_{x,n} \circ \Delta^m_{\alpha,n}.$$

8) *For any function* $f \in \mathcal{L}^1_{\alpha,n}$ *and any bounded continuous function* $g \in \mathscr{C}_b(\mathbb{R})$*, the following integral holds :*

$$\int_0^{+\infty} [T^{\alpha,m}_{x,n} f(y)][e^{-i\frac{a}{b}y^2} g(y)] y^{2\alpha+1} dy = \int_0^{\infty} [e^{-i\frac{a}{b}y^2} f(y)][T^{\alpha,m}_{x,n} g(y)] y^{2\alpha+1} dy. \tag{3.5}$$

9) *For any function* $f \in \mathcal{L}^1_{\alpha,n}$*, we have :*

$$\mathcal{F}^m_{\alpha,n}\left[\overline{T^{\alpha,m}_{x,n}} f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\overline{K^m_{\alpha,n}(\lambda,x)}\mathcal{F}^m_{\alpha,n} f(\lambda). \tag{3.6}$$

10) *For any function* $f \in \mathcal{L}^p_{\alpha,n}$ $p \in]1,2]$ *we have :*

$$\mathcal{F}^m_{\alpha,n}\left[\overline{T^{\alpha,m}_{x,n}} f\right](\lambda) = e^{\frac{i}{2}\frac{d}{b}\lambda^2}\overline{K^m_{\alpha,n}(\lambda,x)}\mathcal{F}^m_{\alpha,n} f(\lambda).$$

Proof: The proof relies on the intertwining relation between the operators $T^{\alpha,m}x,n$ and $T^{\alpha+2n,m}x$ via the mapping $\mathcal{M}$.

1) The identities $T^{\alpha,m}_{0,n} f = f$ and $T^{\alpha,m}_{x,n} f(y) = T^{\alpha,m}_{y,n} f(x)$ follow directly from the corresponding properties of the classical translation $T^{\alpha+2n,m}_x$ and the symmetry of the kernel $W^m_{\alpha+2n}$.

2) Linearity is an immediate consequence of the linearity of $T^{\alpha+2n,m}_x$ and of the operator $\mathcal{M}$.

3) Assume that f vanishes on $[r,+\infty)$. Since $T^{\alpha+2n,m}_x$ preserves compact supports, we obtain $T^{\alpha+2n,m}_x(\mathcal{M}^{-1}f)(y) = 0$ whenever $|x-y| \geq r$, which implies the same property for $T^{\alpha,m}_{x,n} f$.

4) Using Definition (3.2) and the product formula satisfied by $K^m_{\alpha+2n}$, we compute

$$\begin{aligned} T^{\alpha,m}_{x,n}[K^m_{\alpha,n}(y,\cdot)](z) &= (xz)^{2n} T^{\alpha+2n,m}_x [K^m_{\alpha+2n}(y,\cdot)](z) \\ &= (xz)^{2n} e^{-\frac{i}{2}\frac{d}{b}y^2} K^m_{\alpha+2n}(y,x) K^m_{\alpha+2n}(y,z), \end{aligned}$$

which yields the desired identity.

5) The integral representation follows from the corresponding formula for $T^{\alpha+2n,m}_x$ and from the definition of the kernel $W^m_{\alpha,n}$.

6) This follows from the definition of $T^{\alpha,m}_{x,n}$, and the fact that $T^{\alpha,m}_x$ is continuous from $\mathscr{C}(\mathbb{R})$ into itself, $\mathscr{C}_*(\mathbb{R})$ into itself and $\mathcal{L}_{p,\alpha}$ into itself respectively.

   Now let $f \in \mathcal{L}^p_{\alpha,n}$, the function $T^{\alpha,m}_{x,n} f$ belongs to $\mathcal{L}^p_{\alpha,n}$ and we have :

$$\begin{aligned} \|T^{\alpha,m}_{x,n} f\|_{p,\alpha,n} &= x^{2n}\|\mathcal{M} \circ T^{\alpha+2n,m}_x \circ \mathcal{M}^{-1} f\|_{p,\alpha,n} \\ &= x^{2n}\|T^{\alpha+2n,m}_x \circ \mathcal{M}^{-1} f\|_{p,\alpha+2n} \\ &\leqslant x^{2n}\|\mathcal{M}^{-1} f\|_{p,\alpha+2n} \\ &\leqslant x^{2n}\|f\|_{p,\alpha,n}. \end{aligned}$$

7) • Let $f \in \mathscr{C}(\mathbb{R})$ we have :

$$\begin{aligned}\left[T^{\alpha,m}_{x,n} \circ T^{\alpha,m}_{y,n}\right] f(z) &= (xyz)^{2n}\left[T^{\alpha+2n,m}_{x} \circ T^{\alpha+2n,m}_{y}\right](\mathcal{M}^{-1}f)(z)\\ &= (xyz)^{2n}\left[T^{\alpha+2n,m}_{y} \circ T^{\alpha+2n,m}_{x}\right](\mathcal{M}^{-1}f)(z)\\ &= \left[T^{\alpha,m}_{y,n} \circ T^{\alpha,m}_{x,n}\right] f(z).\end{aligned}$$

• Let $f \in \mathscr{C}_*(\mathbb{R})$ then :

$$\begin{aligned}\left[\Delta^{m}_{\alpha,n} \circ T^{\alpha,m}_{x,n}\right] f(y) &= (xy)^{2n}\left[\Delta^{m}_{\alpha+2n} \circ T^{\alpha+2n,m}_{x}\right](\mathcal{M}^{-1}f)(y)\\ &= (xy)^{2n}\left[T^{\alpha+2n,m}_{x} \circ \Delta^{m}_{\alpha+2n}\right](\mathcal{M}^{-1}f)(y)\\ &= T^{\alpha,m}_{x,n} \circ \Delta^{m}_{\alpha,n}.\end{aligned}$$

8) Let $f \in \mathcal{L}^1_{\alpha,n}$ and $g \in \mathscr{C}_b(\mathbb{R})$. Then :

$$\begin{aligned}&\int_0^{+\infty} [T^{\alpha,m}_{x,n} f(y)][e^{-i\frac{a}{b}y^2} g(y)] y^{2\alpha+1} dy\\ &= x^{2n}\int_0^{+\infty} [T^{\alpha+2n,m}_{x}(\mathcal{M}^{-1}f)(y)][e^{-i\frac{a}{b}y^2}(\mathcal{M}^{-1}g)(y)] y^{2\alpha+4n+1} dy\\ &= x^{2n}\int_0^{+\infty} [e^{-i\frac{a}{b}y^2}(\mathcal{M}^{-1}f)(y)][T^{\alpha+2n,m}_{x}(\mathcal{M}^{-1}g)(y)] y^{2\alpha+4n+1} dy\\ &= \int_0^{+\infty} [e^{-i\frac{a}{b}y^2} f(y)][T^{\alpha,m}_{x,n} g(y)] y^{2\alpha+1} dy.\end{aligned}$$

9) Let $f \in \mathcal{L}^1_{\alpha,n}$, then :

$$\begin{aligned}\frac{(ib)^{\alpha+2n+1}}{C_{\alpha+2n}}\mathcal{F}^m_{\alpha,n}\left[\overline{T^{\alpha,m}_{x,n}} f\right](\lambda) &= \int_0^{+\infty} K^m_{\alpha,n}(\lambda,y)\overline{T^{\alpha,m}_{x,n}} f(y) y^{2\alpha+1} dy\\ &= x^{2n} e^{\frac{i}{2}(\frac{d}{b}\lambda^2 - \frac{a}{b}x^2)}\int_0^{+\infty} j_{\alpha+2n}(\frac{\lambda y}{b})\overline{T^{\alpha+2n}_{x}}[e^{\frac{i}{2}\frac{a}{b}s^2}(\mathcal{M}^{-1}f)(s)](y) y^{2\alpha+4n+1} dy\\ &= x^{2n} e^{\frac{i}{2}(\frac{d}{b}\lambda^2 - \frac{a}{b}x^2)}\int_0^{+\infty} y^{2n} e^{\frac{i}{2}\frac{a}{b}y^2} f(y)\overline{T^{\alpha+2n}_{x}}[s \to j_{\alpha+2n}(\frac{\lambda s}{b})](y) y^{2\alpha+1} dy\\ &= x^{2n} e^{-\frac{i}{2}\frac{a}{b}x^2} j_{\alpha+2n}(\frac{\lambda x}{b})\int_0^{+\infty} K^m_{\alpha,n}(\lambda,y) f(y) y^{2\alpha+1} dy\\ &= \frac{(ib)^{\alpha+2n+1}}{C_{\alpha+2n}} e^{\frac{i}{2}\frac{d}{b}\lambda^2}\overline{K^m_{\alpha,n}(\lambda,x)}\mathcal{F}^m_{\alpha,n} f(\lambda).\end{aligned}$$

10) From (9) the result is true for $f \in \mathcal{L}^1_{\alpha,n} \cap \mathcal{L}^p_{\alpha,n}$. On the other hand Babenko inequality (**??**) and the relation (3.4) show that the mappings $f \to \mathcal{F}^m_{\alpha,n}\left[\overline{T^{\alpha,m}_{x,n}} f\right]$ and $f \to \mathcal{F}^m_{\alpha,n} f$ are continuous from $\mathcal{L}^p_{\alpha,n}$ into $\mathcal{L}^q_{\alpha,n}$ $(\frac{1}{p} + \frac{1}{q} = 1)$. We obtain the result from density of $\mathcal{L}^1_{\alpha,n} \cap \mathcal{L}^p_{\alpha,n}$ in $\mathcal{L}^p_{\alpha,n}$.

□

Corollary 3.1 *Let $m \in SL(2,\mathbb{R})$ such that $b \neq 0$ and $x \in \mathbb{R}$. Then the operator $\overline{T^{\alpha,m}_{x,n}}$ leaves $\mathcal{S}_*(\mathbb{R})$ invariant, and for any $f \in \mathcal{S}_*(\mathbb{R})$ we have :*

$$\overline{T^{\alpha,m}_{x,n}} f(y) = \frac{C_{\alpha+2n}}{(-ib)^{\alpha+2n+1}} e^{-\frac{i}{2}\frac{a}{b}y^2}\int_0^{+\infty} j_{\alpha+2n}\left(\frac{\lambda y}{b}\right)\overline{K^m_{\alpha,n}(\lambda,x)}\mathcal{F}^m_{\alpha,n} f(\lambda)\lambda^{2\alpha+2n+1} d\lambda.$$

Proof: Observe that if $f \in \mathcal{S}_*(\mathbb{R})$ then the inequality (3.4) shows that $y \to \overline{[T^{\alpha,m}_{x,n} f]}(y)$ is a continuous function of $\mathcal{L}^1_{\alpha,n}$. The result is then a consequence of (3.6) and the reversibility property of the generalized canonical Fourier-Bessel transform. $\square$

Theorem 3.1 *Let $m \in SL(2,\mathbb{R})$ such that $b \neq 0$.*

1) *For any $f \in \mathscr{C}_{*,0}(\mathbb{R})$, we have $\lim_{y\to+\infty} \|T^{\alpha,m}_{y,n} f - f\|_\infty = 0$.*

2) *For any $f \in \mathcal{L}^p_{\alpha,n}$, with $1 \leqslant p < \infty$, we have $\lim_{y\to+\infty} \|T^{\alpha,m}_{y,n} f - f\|_{p,\alpha,n} = 0$.*

Proof:

1) Let $f \in \mathscr{C}_{*,0}(\mathbb{R})$

$$\begin{aligned} T^{\alpha,m}_{y,n} f(x) - f(x) &= (xy)^{2n} T^{\alpha+2n,m}_y (\mathcal{M}^{-1} f)(x) - x^{2n} (\mathcal{M}^{-1} f)(x) \\ &= x^{2n} \left[y^{2n} T^{\alpha+2n,m}_y (\mathcal{M}^{-1} f)(x) - (\mathcal{M}^{-1} f)(x) \right], \end{aligned}$$

so

$$\|T^{\alpha,m}_{y,n} f - f\|_\infty \leqslant x^{2n} \|T^{\alpha+2n,m}_y (\mathcal{M}^{-1} f) - \mathcal{M}^{-1} f\|_\infty$$

hence $$\lim_{y\to+\infty} \|T^{\alpha,m}_{y,n} f - f\|_\infty = 0.$$

2) Follows from (1) and density of $\mathscr{C}_{*,0}(\mathbb{R})$ in $\mathcal{L}^p_{\alpha,n}$.

$\square$

4. Generalized convolution product

Let $m \in SL(2,\mathbb{R})$ with $b \neq 0$. Given two measurable functions f and g on $[0,+\infty)$, we define their generalized convolution associated with $\Delta^m_{\alpha,n}$ by

$$(f \underset{\alpha,m,n}{*} g)(x) = \int_0^\infty \left[T^{\alpha,m}_{x,n} f\right](y) \left[e^{-i\frac{a}{b}y^2} g(y)\right] y^{2\alpha+1}\, dy, \tag{4.1}$$

whenever the integral is well defined.

Proposition 4.1 *Assume that all integrals converge. The convolution $\underset{\alpha,m,n}{*}$ satisfies:*

(i) ***Commutativity.***

$$f \underset{\alpha,m,n}{*} g = g \underset{\alpha,m,n}{*} f.$$

(ii) ***Translation invariance.*** *For all $x \geq 0$,*

$$T^{\alpha,m}_{x,n}(f \underset{\alpha,m,n}{*} g) = (T^{\alpha,m}_{x,n} f) \underset{\alpha,m,n}{*} g = f \underset{\alpha,m,n}{*} (T^{\alpha,m}_{x,n} g).$$

(iii) ***Associativity.***

$$f \underset{\alpha,m,n}{*} (g \underset{\alpha,m,n}{*} h) = (f \underset{\alpha,m,n}{*} g) \underset{\alpha,m,n}{*} h.$$

Proof:

1) This follows directly from the symmetry of the translation operator, but we give here another proof. From Fubini's Theorem :

$$\begin{aligned} f \underset{\alpha,m,n}{*} g(x) &= \int_0^{+\infty} \left[T_{x,n}^{\alpha,m} f\right](y) \left[e^{-i\frac{a}{b}y^2} g(y)\right] y^{2\alpha+1} dy \\ &= \int_0^{+\infty} \left[\int_0^{+\infty} e^{-i\frac{a}{b}z^2} f(z) W_{\alpha,n}^m(x,y,z) z^{2\alpha+1} dz\right] \left[e^{-i\frac{a}{b}y^2} g(y)\right] y^{2\alpha+1} dy \\ &= \int_0^{+\infty} \left[\int_0^{+\infty} e^{-i\frac{a}{b}y^2} g(y) W_{\alpha,n}^m(x,y,z) y^{2\alpha+1} dy\right] \left[e^{-i\frac{a}{b}z^2} f(z)\right] z^{2\alpha+1} dz \\ &= g \underset{\alpha,m,n}{*} f(x). \end{aligned}$$

2. By Fubini's Theorem :

$$\begin{aligned} T_{x,n}^{\alpha,m}\left(f \underset{\alpha,m,n}{*} g\right)(y) &= \int_0^{+\infty} e^{-i\frac{a}{b}z^2} \left(f \underset{\alpha,m,n}{*} g\right)(z) W_{\alpha,n}^m(x,y,z) z^{2\alpha+1} dz \\ &= \int_0^{+\infty} e^{-i\frac{a}{b}z^2} \left[\int_0^{+\infty} \left[T_{z,n}^{\alpha,m} f(s)\right] \left[e^{-i\frac{a}{b}s^2} g(s)\right] s^{2\alpha+1} ds\right] W_{\alpha,n}^m(x,y,z) z^{2\alpha+1} dz \\ &= \int_0^{+\infty} \left[\int_0^{+\infty} e^{-i\frac{a}{b}z^2} \left[T_{s,n}^{\alpha,m} f(z)\right] W_{\alpha,n}^m(x,y,z) z^{2\alpha+1} dz\right] \left[e^{-i\frac{a}{b}s^2} g(s)\right] s^{2\alpha+1} ds \\ &= \int_0^{+\infty} T_{x,n}^{\alpha,m} \left[T_{s,n}^{\alpha,m} f\right](y) \left[e^{-i\frac{a}{b}s^2} g(s)\right] s^{2\alpha+1} ds \\ &= \int_0^{+\infty} T_{y,n}^{\alpha,m} \left[T_{x,n}^{\alpha,m} f\right](s) \left[e^{-i\frac{a}{b}s^2} g(s)\right] s^{2\alpha+1} ds \\ &= \left(\left[T_{x,n}^{\alpha,m} f\right] \underset{\alpha,m,n}{*} g\right)(y). \end{aligned}$$

3) Follows (1), (2) and Fubini's Theorem :

$$\begin{aligned} \left(\left[f \underset{\alpha,m,n}{*} g\right] \underset{\alpha,m,n}{*} h\right)(x) &= \int_0^{+\infty} T_{x,n}^{\alpha,m} \left[f \underset{\alpha,m,n}{*} g\right](y) \left[e^{-i\frac{a}{b}y^2} h(y)\right] y^{2\alpha+1} dy \\ &= \int_0^{+\infty} \left[g \underset{\alpha,m,n}{*} T_{x,n}^{\alpha,m} f\right](y) \left[e^{-i\frac{a}{b}y^2} h(y)\right] y^{2\alpha+1} dy \\ &= \int_0^{+\infty} \left[\int_0^{+\infty} T_{y,n}^{\alpha,m} g(s) e^{-i\frac{a}{b}s^2} T_{x,n}^{\alpha,m} f(s) s^{2\alpha+1} ds\right] \left[e^{-i\frac{a}{b}y^2} h(y)\right] y^{2\alpha+1} dy \\ &= \int_0^{+\infty} T_{x,n}^{\alpha,m} f(s) e^{-i\frac{a}{b}s^2} \left[\int_0^{+\infty} T_{s,n}^{\alpha,m} g(y) \left[e^{-i\frac{a}{b}y^2} h(y)\right] y^{2\alpha+1} dy\right] s^{2\alpha+1} ds \\ &= \int_0^{+\infty} T_{x,n}^{\alpha,m} f(s) e^{-i\frac{a}{b}s^2} \left[g \underset{\alpha,m,n}{*} h\right](s)^{2\alpha+1} ds \\ &= \left(f \underset{\alpha,m,n}{*} \left[g \underset{\alpha,m,n}{*} h\right]\right)(x). \end{aligned}$$

□

Remark 4.1 It is easy to see that :

$$f \underset{\alpha,m,n}{*} g = \mathcal{M}\left[(\mathcal{M}^{-1} f) \underset{\alpha+2n,m}{*} (\mathcal{M}^{-1} g)\right], \tag{4.2}$$

where $\underset{\alpha+2n,m}{*}$ is the convolution product given in (**??**).

Proposition 4.2 (Young's inequality) *Let $m \in SL(2,\mathbb{R})$ such that $b \neq 0$.*
Suppose $1 \leqslant p,\ q,\ r \leqslant +\infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$.
If $f \in \mathcal{L}^p_{\alpha,n}$ satisfy $g \in \mathcal{L}^q_{\alpha,n}$ then $f \underset{\alpha,m,n}{*} g \in \mathcal{L}^r_{\alpha,n}$ and we have :

$$\|f \underset{\alpha,m,n}{*} g\|_{r,\alpha,n} \leqslant \|f\|_{p,\alpha,n}\|g\|_{q,\alpha,n}. \tag{4.3}$$

Proof: Using (4.2)

$$\begin{aligned}
\|f \underset{\alpha,m,n}{*} g\|_{r,\alpha,n} &= \|(\mathcal{M}^{-1}f) \underset{\alpha+2n,m}{*} (\mathcal{M}^{-1}g)\|_{r,\alpha+2n} \\
&\leqslant \|\mathcal{M}^{-1}f\|_{p,\alpha+2n}\|\mathcal{M}^{-1}g\|_{q,\alpha+2n} \\
&= \|f\|_{p,\alpha,n}\|g\|_{q,\alpha,n}.
\end{aligned}$$

□

Proposition 4.3 *Let $m \in SL(2,\mathbb{R})$ such that $b \neq 0$.*

1) Let f and g be two functions in $\mathcal{L}^1_{\alpha,n}$. We have :

$$\forall x \in \mathbb{R}, \qquad \frac{C_{\alpha+2n}}{(ib)^{\alpha+2n+1}}\mathcal{F}^m_{\alpha,n}\left(\overline{f \underset{\alpha,m,n}{*} g}\right)(x) = e^{-\frac{i}{2}\frac{d}{b}x^2}\mathcal{F}^m_{\alpha,n}f(x)\mathcal{F}^m_{\alpha,n}g(x).$$

2) Let $f \in \mathcal{L}^1_{\alpha,n}$ and $g \in \mathcal{L}^p_{\alpha,n}$ $(p \in]1,2])$. We have :

$$\frac{C_{\alpha+2n}}{(ib)^{\alpha+2n+1}}\mathcal{F}^m_{\alpha,n}\left(\overline{f \underset{\alpha,m,n}{*} g}\right)(x) = e^{-\frac{i}{2}\frac{d}{b}x^2}\mathcal{F}^m_{\alpha,n}f(x)\mathcal{F}^m_{\alpha,n}g(x).$$

Proof:

1) According to the previous proposition, one has $\overline{f \underset{\alpha,m,n}{*} g} \in \mathcal{L}^1_{\alpha,n}$. From the definition of $\mathcal{F}^m_{\alpha,n}$ and (3.6) it is easy to see that

$$\begin{aligned}
\mathcal{F}^m_{\alpha,n}\left(\overline{f \underset{\alpha,m,n}{*} g}\right)(x) &= \frac{C_{\alpha+2n}}{(ib)^{\alpha+2n+1}}\int_0^{+\infty} K^m_{\alpha,n}(x,y)\left(\overline{f \underset{\alpha,m,n}{*} g}\right)(y)y^{2\alpha+1}dy \\
&= \frac{C_{\alpha+2n}}{(ib)^{\alpha+2n+1}}\int_0^{+\infty} K^m_{\alpha,n}(x,y)\left[\int_0^{+\infty}\overline{T^{\alpha,m}_{y,n}}f(z)\left[e^{i\frac{a}{b}z^2}g(z)\right]z^{2\alpha+1}dz\right]y^{2\alpha+1}dy \\
&= \frac{C_{\alpha+2n}}{(ib)^{\alpha+2n+1}}\int_0^{+\infty} e^{i\frac{a}{b}z^2}g(z)\left[\int_0^{+\infty} K^m_{\alpha,n}(x,y)\overline{T^{\alpha,m}_{z,n}}f(y)y^{2\alpha+1}dy\right]z^{2\alpha+1}dz \\
&= \int_0^{+\infty} e^{i\frac{a}{b}z^2}g(z)\left[\mathcal{F}^m_{\alpha,n}\left(\overline{T^{\alpha,m}_{z,n}}f\right)(x)\right]z^{2\alpha+1}dz \\
&= \frac{(ib)^{\alpha+2n+1}}{C_{\alpha+2n}}e^{-\frac{i}{2}\frac{d}{b}x^2}\mathcal{F}^m_{\alpha,n}f(x)\mathcal{F}^m_{\alpha,n}g(x).
\end{aligned}$$

2) From (1) the result is true for $g \in \mathcal{L}^1_{\alpha,n} \cap \mathcal{L}^p_{\alpha,n}$. . On the other hand, the Babenko inequality and Proposition (4.2) show that the mappings $f \to \mathcal{F}^m_{\alpha,n}\left(\overline{f \underset{\alpha,m,n}{*} g}\right)$ and $g \to \mathcal{F}^m_{\alpha,n}f\mathcal{F}^m_{\alpha,n}g$ are continuous from $\mathcal{L}^p_{\alpha,n}$ into $\mathcal{L}^q_{\alpha,n}$ $(\frac{1}{p} + \frac{1}{q} = 1)$. We obtain the result from density of $\mathcal{L}^1_{\alpha,n} \cap \mathcal{L}^p_{\alpha,n}$ in $\mathcal{L}^p_{\alpha,n}$.

□

5. Heat Equation Associated with $\overline{\Delta^m_{\alpha,n}}$

In this section, we study the heat equation associated with the conjugate of the generalized Bessel-type operator $\Delta^m_{\alpha,n}$ using the generalized translation operator and the generalized convolution product. Our goal is to construct explicit solutions by means of the generalized linear canonical Fourier-Bessel transform, building on the properties developed in earlier sections.

We consider the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \sigma\, \overline{\Delta^m_{\alpha,n}}\, u(t,x), & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{5.1}$$

where $f \in \mathcal{L}^p_{\alpha,n}$ with $1 \leqslant p \leqslant \infty$, and $\sigma > 0$ is the heat conductivity. The initial condition means that $u(t,x) \to f(x)$ as $t \to 0$ in the $\mathcal{L}^p_{\alpha,n}$-norm.

Formal Solution Using the Transform

Applying the generalized linear canonical Fourier-Bessel transform $\mathcal{F}^m_{\alpha,n}$, the PDE reduces to the ordinary differential equation

$$\left(\partial_t + \frac{\sigma}{b^2}x^2\right)\mathcal{F}^m_{\alpha,n}[u(t,\cdot)](x) = 0.$$

Its general solution is

$$\mathcal{F}^m_{\alpha,n}[u(t,\cdot)](x) = c(x)\, e^{-\frac{\sigma t}{b^2}x^2}.$$

By imposing the initial condition $\mathcal{F}^m_{\alpha,n}[u(0,\cdot)](x) = \mathcal{F}^m_{\alpha,n}[f](x)$, we obtain

$$c(x) = \mathcal{F}^m_{\alpha,n}[f](x), \quad \text{hence} \quad \mathcal{F}^m_{\alpha,n}[u(t,\cdot)](x) = e^{-\frac{\sigma t}{b^2}x^2}\, \mathcal{F}^m_{\alpha,n}[f](x).$$

Using the integral identity [8]

$$\int_0^{+\infty} e^{-\beta y^2} j_{\alpha+2n}\left(\frac{xy}{b}\right) y^{2\alpha+4n+1} dy = \frac{\Gamma(\alpha+2n+1)}{2\beta^{\alpha+2n+1}} e^{-\frac{x^2}{4\beta b^2}}, \quad \beta > 0,\ b \neq 0,$$

we can write

$$e^{-\frac{\sigma t}{b^2}x^2} = \frac{(ib)^{\alpha+2n+1}}{C_{\alpha+2n}} e^{-\frac{i}{2}\frac{d}{b}x^2} \mathcal{F}^m_{\alpha,n}[\mathcal{P}^m_{t,n}](x),$$

where

$$\mathcal{P}^m_{t,n}(x) = \frac{2}{\Gamma(\alpha+2n+1)}(4\sigma t)^{-(\alpha+2n+1)} x^{2n} e^{-\frac{i}{2}\frac{a}{b}x^2 - \frac{x^2}{4\sigma t}}.$$

By Proposition 4.3, it follows that

$$\mathcal{F}^m_{\alpha,n}[u(t,\cdot)](x) = \mathcal{F}^m_{\alpha,n}\left(\overline{\mathcal{P}^m_{t,n} \underset{\alpha,m,n}{*} f}\right)(x),$$

which leads to the candidate solution

$$u(t,x) = \overline{\mathcal{P}^m_{t,n} \underset{\alpha,m,n}{*} f}(x) = \int_0^{+\infty} \overline{T^{\alpha,m} x,n}[\mathcal{P}^m t,n](y)\, e^{i\frac{a}{b}y^2} f(y)\, y^{2\alpha+1} dy.$$

5.1. Generalized Heat Kernel

Definition 5.1 *Let $m \in SL(2,\mathbb{R})$ with $b \neq 0$. The* generalized heat kernel $G^m_{t,n}$ *is defined by*

$$G^m_{t,n}(x,y) := \overline{T^{\alpha,m} x,n}[\mathcal{P}^m t,n](y), \quad x,y \in \mathbb{R},\ t > 0. \tag{5.2}$$

Lemma 5.1 (Bessel Product Formula [8]) *Let $B, r, s \in \mathbb{C}$ with $\Re(B) > 0$. Then*

$$\int_0^{+\infty} e^{-Bx^2} j_{\alpha+2n}(2rx) j_{\alpha+2n}(2sx) x^{2\alpha+4n+1} dx = \frac{\Gamma(\alpha+2n+1)}{2B^{\alpha+2n+1}} e^{-\frac{r^2+s^2}{B}} j_{\alpha+2n}\left(\frac{2irs}{B}\right). \tag{5.3}$$

5.2. Properties of the Generalized Heat Kernel

Proposition 5.1 *Let $m \in SL(2,\mathbb{R})$ with $b \neq 0$. Then $G^m_{t,n}$ satisfies the following properties:*

1. Explicit formula:
$$G^m_{t,n}(x,y) = \frac{2}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}} e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)} e^{-\frac{x^2+y^2}{4\sigma t}} (xy)^{2n} j_{\alpha+2n}\Big(\frac{ixy}{2\sigma t}\Big).$$

2. Upper bound:
$$|G^m_{t,n}(x,y)| \leq \frac{2(xy)^{2n}}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}} e^{-\frac{(|x|-|y|)^2}{4\sigma t}}.$$

3. Reproducing property:
$$\int_0^{+\infty} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G^m_{t,n}(x,y) y^{2\alpha+2n+1} dy = x^{2n}.$$

4. Semigroup property:
$$G^m_{t+s,n}(x,y) = \int_0^{+\infty} e^{i\frac{a}{b}z^2} G^m_{t,n}(x,z) G^m_{s,n}(y,z) z^{2\alpha+1} dz.$$

5. *For fixed $y \in \mathbb{R}$, the function $u(t,x) = G^m_{t,n}(x,y)$ solves the heat equation:*
$$\partial_t u(t,x) = \sigma \overline{\Delta^m_{\alpha,n}} u(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}.$$

Proof:

1) From the definition of $T^{\alpha,m}_{x,n}$ it follows that
$$\begin{aligned} G^m_{t,n}(x,y) &= \overline{T^{\alpha,m}_{x,n}}[\mathcal{P}^m_{t,n}](y) \\ &= (xy)^{2n} \overline{T^{\alpha+2n,m}_x}[\mathcal{M}^{-1}\mathcal{P}^m_{t,n}](y) \\ &= \frac{2e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)} e^{-\frac{y^2}{4\sigma t}}}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}} (xy)^{2n} \overline{T^{\alpha+2n}_x}[e^{-\frac{x^2}{4\sigma t}}](y). \end{aligned}$$

A simple calculation show that
$$\begin{aligned} \overline{T^{\alpha+2n}_x}[e^{-\frac{x^2}{4\sigma t}}](y) &= \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)} \int_0^\pi e^{-\frac{x^2+y^2-2xy\cos\theta}{4\sigma t}} (\sin\theta)^{2\alpha+4n} d\theta \\ &= \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)} e^{-\frac{x^2+y^2}{4\sigma t}} \int_0^\pi e^{\frac{xy\cos\theta}{2\sigma t}} (\sin\theta)^{2\alpha+4n} d\theta \\ &= \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)} e^{-\frac{x^2+y^2}{4\sigma t}} \sum_{k=0}^{+\infty} \frac{(\frac{xy}{2\sigma t})^k}{k!} \int_0^\pi \cos^k(\theta)(\sin(\theta))^{2\alpha+4n} d\theta. \end{aligned}$$

By the equation (3.7) in [8], we deduce
$$\overline{T^{\alpha+2n}_x}[e^{-\frac{x^2}{4\sigma t}}](y) = e^{-\frac{x^2+y^2}{4\sigma t}} j_{\alpha+2n}(\frac{ixy}{2\sigma t}).$$

3) From Lemma 5.1 we know that
$$\begin{aligned} &\int_0^{+\infty} e^{\frac{i}{2}\frac{a}{b}(x^2+y^2)} G^m_{t,n}(x,y) y^{2\alpha+2n+1} dy \\ &= \frac{2e^{-\frac{x^2}{4\sigma t}}}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}} x^{2n} \int_0^{+\infty} e^{-\frac{y^2}{4\sigma t}} j_{\alpha+2n}(\frac{ixy}{2\sigma t}) y^{2\alpha+4n+1} dy = x^{2n}. \end{aligned}$$

4) We have :

$$\int_0^{+\infty} e^{i\frac{a}{b}z^2} G^m_{t,n}(x,z)G^m_{s,n}(y,z)z^{2\alpha+1}dz$$
$$= \frac{4e^{-\frac{x^2}{4\sigma t}}e^{-\frac{y^2}{4\sigma s}}e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)}}{\Gamma^2(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}(4\sigma s)^{\alpha+2n+1}} \times \int_0^{+\infty} e^{-\frac{z^2}{4\sigma t}}e^{-\frac{z^2}{4\sigma s}} j_{\alpha+2n}(\frac{ixz}{2\sigma t})j_{\alpha+2n}(\frac{iyz}{2\sigma s})z^{2\alpha+4n+1}dz. \tag{5.4}$$

By Lemma 5.1 we have

$$\int_0^{+\infty} e^{-\frac{z^2}{4\sigma}\frac{t+s}{ts}} j_{\alpha+2n}(\frac{ixz}{2\sigma t})j_{\alpha+2n}(\frac{iyz}{2\sigma s})z^{2\alpha+4n+1}dz$$
$$= \frac{\Gamma(\alpha+2n+1)(4\sigma ts)^{\alpha+2n+1}}{2(t+s)^{\alpha+2n+1}} e^{\frac{sx^2}{4\sigma t(t+s)}} e^{\frac{ty^2}{4\sigma s(t+s)}} j_{\alpha+2n}(\frac{ixy}{2\sigma(t+s)}).$$

Now a simple calculation shows that

$$\int_0^{+\infty} e^{i\frac{a}{b}z^2} G^m_{t,n}(x,z)G^m_{s,n}(y,z)z^{2\alpha+1}dz$$
$$= \frac{2}{\Gamma(\alpha+2n+1)(4\sigma(t+s))^{\alpha+2n+1}} \times e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)}e^{-\frac{x^2+y^2}{4\sigma(t+s)}}(xy)^{2n} j_{\alpha+2n}(\frac{ixy}{2\sigma(t+s)})$$
$$= G^m_{t+s,n}(x,y).$$

5) For $y \in \mathbb{R}$ and $t > 0$

$$\frac{\partial}{\partial t}G^m_{t,n}(x,y) = \left(\frac{x^2+y^2}{4\sigma t^2} - \frac{\alpha+2n+1}{t}\right)G^m_{t,n}(x,y) - \frac{2e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)}e^{-\frac{x^2+y^2}{4\sigma t}}}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}}\left(-\frac{ixy}{2\sigma t^2}\right)j'_{\alpha+2n}(\frac{ixy}{2\sigma t}). \tag{5.5}$$

By the transmutations property

$$\mathcal{M}^{-1} \circ \overline{\Delta^m_{\alpha,n}} \circ \mathcal{M} = \overline{\Delta^m_{\alpha+2n}}, \tag{5.6}$$

and

$$e^{\frac{i}{2}\frac{a}{b}x^2}\overline{\Delta^m_{\alpha+2n}} \circ e^{-\frac{i}{2}\frac{a}{b}x^2} = \overline{\Delta^m_{\alpha+2n}}. \tag{5.7}$$

We have

$$\sigma\overline{\Delta^m_{\alpha,n}}\left[G^m_{t,n}(x,y)\right] = \sigma\mathcal{M} \circ \overline{\Delta^m_{\alpha+2n}}\left[\mathcal{M}^{-1}G^m_{t,n}(x,y)\right]$$
$$= \sigma\mathcal{M}\left[e^{-\frac{i}{2}\frac{a}{b}x^2}\overline{\Delta_{\alpha+2n}}\left(e^{\frac{i}{2}\frac{a}{b}x^2}\mathcal{M}^{-1}G^m_{t,n}(x,y)\right)\right]$$
$$= \frac{2\sigma e^{-\frac{i}{2}\frac{a}{b}(x^2+y^2)}e^{-\frac{y^2}{4\sigma t}}}{\Gamma(\alpha+2n+1)(4\sigma t)^{\alpha+2n+1}}(xy)^{2n}\overline{\Delta_{\alpha+2n}}\left[e^{-\frac{x^2}{4\sigma t}}j_{\alpha+2n}(\frac{ixy}{2\sigma t})\right].$$

An easy calculation shows that

$$\overline{\Delta_{\alpha+2n}}\left[e^{-\frac{x^2}{4\sigma t}}j_{\alpha+2n}(\frac{ixy}{2\sigma t})\right]$$
$$= e^{-\frac{x^2}{4\sigma t}}\left[(\frac{iy}{2\sigma t})^2 j''_{\alpha+2n}(\frac{ixy}{2\sigma t}) + (\frac{iy(2\alpha+4n+1)}{2\sigma tx} - \frac{ixy}{2\sigma^2t^2})j'_{\alpha+2n}(\frac{ixy}{2\sigma t})\right. \tag{5.8}$$
$$\left.+(\frac{x^2}{4\sigma^2t^2} - \frac{\alpha+2n+1}{\sigma t})j_{\alpha+2n}(\frac{ixy}{2\sigma t})\right]. \tag{5.9}$$

Since $j''_{\alpha+2n}(z) + \dfrac{2\alpha+4n+1}{z} j'_{\alpha+2n}(z) = -j_{\alpha+2n}(z)\ z \in \mathbb{C}$.

We denote

$$\overline{\Delta_{\alpha+2n}}\left[e^{-\frac{x^2}{4\sigma t}} j_{\alpha+2n}(\frac{ixy}{2\sigma t})\right] = e^{-\frac{x^2}{4\sigma t}}\left[\left(\frac{x^2+y^2}{4\sigma^2 t^2} - \frac{\alpha+2n+1}{\sigma t}\right) j_{\alpha+2n}(\frac{ixy}{2\sigma t}) - \frac{ixy}{2\sigma^2 t^2} j'_{\alpha+2n}(\frac{ixy}{2\sigma t})\right].$$

Then

$$\frac{\partial}{\partial t} u(t,x) = \sigma \overline{\Delta^m_{\alpha,n}} u(t,x).$$

□

5.3. Solution of the Cauchy Problem

Proposition 5.2 *Let $m \in SL(2,\mathbb{R})$ with $b \neq 0$, and let $f \in \mathcal{L}^p_{\alpha,n}$, $1 \leq p \leq \infty$. Define*

$$u(t,x) := \overline{\mathcal{P}^m_{t,n} \underset{\alpha,m,n}{*} f}(x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}.$$

Then:

1. *u satisfies the heat equation (5.1).*
2. *The following estimate holds:*

$$\|u(t,\cdot)\|r,\alpha,n \leq \left[\frac{(4\sigma t)^{\alpha+2n+1}\Gamma(\alpha+2n+1)}{2}\right]^{\frac{1}{q}-1} q^{-(\alpha+2n+1)/q} \|f\|p,\alpha,n,$$

   where $p,q,r \in [1,\infty]$ satisfy $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$.

3. *$u(t,\cdot)$ is the unique solution of (5.1) in $\mathcal{L}^r_{\alpha,n}$ for $1 \leq r \leq \infty$.*

Proof (Sketch). The result follows from expressing $u(t,x)$ as a convolution with the generalized heat kernel:

$$u(t,x) = \int_0^{+\infty} G^m_{t,n}(x,y)\, e^{i\frac{a}{b}y^2} f(y)\, y^{2\alpha+1} dy,$$

applying the semigroup property (Proposition 5.1) and Young's inequality for $\mathcal{L}^p_{\alpha,n}$ spaces, and using standard uniqueness arguments. The kernel $G^m_{t,n}$ satisfies the heat equation, which ensures that u solves the PDE.

References

1. Akhlidj A., Z. SADIK, Uncertainty principles for the generalized linear canonical Fourier-Bessel transform, Journal of Mathematical Sciences, https://doi.org/10.1007/s10958-024-07333-4.
2. Barshan B, Kutay M, Ozaktas H. Optimal filtering with linear canonical transformations. Opt Commun. 1997;135:32–36.
3. Bultheel A., Marlinez-Sulbaran H., Recent development in the theory of the fractional Fourier and linear canonical transforms, Bull. Belg. Math. Soc. Simon Stevin. 13 (2006), no .5, 971-1005.
4. Delsarte J., Sur une extension de la formule de Taylor, Journal de Math. Pur. Appl. 17 (1938), Fasc III, 213-231.
5. Dhaouadi L., Sahbani J. and Fitouhi A., Harmonic analysis associated to the canonical Fourier-Bessel transform, Integral transform and Special Functions, DOI 10.1080/10652469.2020.1823977.
6. Dunkl C. F., Hankel transforms associated to finite reflection groups, Contemp. Math. 138 (1) 128-138 (1992).
7. Ghazouani S. and Bouzeffour F. , A fractional power for Dunkl transform, Bull. Math. Anal. Appl. 6 (3) (2014) 1-30.
8. Ghazouani S., Sahbani J., Generalized translation operator and the Heat equation for the canonical Fourier-Bessel transform, arXiv : 2104.05587v1[math.AP] 12 Apr 2021.
9. Ghazouani S., Soltani E. A. and Fitouhi A., A unified class of integral transforms related to the Dunkl transform, J. Math. Anal. Appl. 449 (2017), no. 2, 1797-1849.

10. Healy J. J., Kutay M. A., H. M. Ozaktas H. M. and Sheridan J. T., Linear Canonical Transforms: Theory and Applications, Springer Series in Optical Sciences 198, Springer, New York 2016.

11. Pei S, Ding J. Eigenfunctions of linear canonical transform. IEEE Trans Signal Process. 2002;50:11–26.

12. Trimèche K., Generalized Harmonic Analysis and Wavelet Packets, Gordon and Breach Science Publishers, 2001.

Zakaria SADIK,
Fundamental and Applied Mathematics Laboratory
Department of Mathematics and Informatics
Faculty of Science Aïn Chock
University Hassan II,
Morocco.
E-mail address: sadzak.095@gmail.com