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Constrained Recurrent Cubic Fractal Framework and Forecasting based on Decision Tree

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ABSTRACT: In this paper, we propose a Constrained Recurrent Cubic Fractal Model that integrates the Recurrent Iterated Function System with a constrained piecewise linear function to model complex datasets and establish its convergence analysis effectively. The proposed model employs rational cubic and quadratic forms $\frac{P_\kappa(\mathbf{s})}{Q_\kappa(\mathbf{s})}$ to ensure precise interpolation. To validate its effectiveness, we apply the model to real-world datasets, including stock data analyzed using decision tree regression. The integration of decision tree regression further enhances predictive performance, enabling accurate interpolation of existing data points and reliable forecasting of future values. Numerical experiments confirm that the model produces smooth and accurate interpolations, preserves underlying trends, and delivers consistent predictions across diverse datasets. This study represents a significant advancement in recurrent fractal-based modeling methodologies for data analysis and forecasting.

Keywords: Recurrent iterated function system, recurrent rational fractal interpolation, convergence, constrained interpolation, forecasting, decision tree regression.

Contents

1. Introduction

The incorporation of iterated function systems (IFS) into interpolation has led to the development of fractal-based approaches capable of outperforming classical techniques when dealing with irregular, highly fluctuating, or self-similar datasets. Due to their recursive construction, fractal interpolation functions can recover fine-scale structures in data that traditional schemes often fail to detect, thereby enabling successful applications in domains such as medical image reconstruction, geophysical surface modelling, and dynamic signal analysis. The concept of IFS, introduced by Barnsley, laid the foundation for the systematic formulation of Fractal Interpolation Functions (FIFs) [7,9]. Later, Barnsley and Harrington proposed $\mathcal{C}^r$–FIFs that require prescribing all derivatives up to order r at the left endpoint of the interpolation interval [8]. Abdulla and co-authors modified Non-Affine Fractal Interpolation (NAFI) by optimizing vertical scaling factors through advanced strategies such as Fractal Nelder–Mead, Fractal Particle Swarm Optimization, and Fractal Differential Evolution, and applied these to forecasting models—ARIMA, Nu-SVR, and neural networks—using Euclidean-distance-based fitness measures [3]. In a subsequent work, Abdulla et al. introduced Fractal Differential Evolution (FDE) to enhance the design of Rational Fractal

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Cubic (RFC) splines, integrating them with artificial neural networks to improve extrapolation behavior [2]. Reddy established smooth zipper FIFs and proved that a regular fractal function corresponds to a certain derivative of these constructions under fixed scaling factors [19]. Chand and collaborators examined constrained rational cubic FIFs, showing that suitable scaling and form parameters allow the generation of interpolants that lie between piecewise linear functions or within prescribed bounds, while maintaining convergence and shape preservation [12,20]. A related C^1 rational cubic scheme by Delbourgo and Gregory ensured monotonicity and convexity while including rigorous error estimates [15]. Chand et al. later generalized rational cubic FIFs via $\frac{p_\kappa(\mathbf{s})}{q_\kappa(\mathbf{s})}$ representations incorporating tension parameters and scaling factors that support positivity, convexity, and monotonicity constraints [13]. With the introduction of an α-fractal rational quartic spline, Chand and colleagues further established a unified C^2-continuous framework for constructing smooth and bounded fractal curves with controllable error behavior [17]. The rational cubic IFS using $\frac{E_\kappa}{F_\kappa}$ forms proposed by Chand et al. expanded the flexibility of classical interpolants by guaranteeing convergence to C^4 originals and permitting control over derivative smoothness [11]. Chand and Vijender developed C^1 rational quadratic FIFs that ensure monotonicity without shape parameters by imposing bounds on scaling factors, enabling both local control and visual smoothness [14]. Katiyar et al. proposed a cubic rational FIF framework with two to three shape parameters and an additional tension factor for improved modeling power [18]. Reddy constructed rational cubic spline FIFs on rectangular domains using partial blending and studied their robustness against perturbations in scaling parameters [24]. Abdulla, Sana, and Mahipal Reddy also demonstrated the usefulness of IFS models in creative digital design through clustering-based segmentation of fashion patterns [1]. Other works by Reddy et al. extended Bézier curves to their fractal counterparts via matrix subdivision while retaining smoothness and shape preservation [25], and Chand et al. examined fractal modifications of Bézier forms in 2D and 3D, establishing contractive and convergence requirements [26]. Drakopoulos et al. constructed affine bivariate FIFs through IFS and derived conditions to ensure positivity and contractivity [16]. Balasubramani et al. introduced α-fractal rational cubic splines with boundedness and positivity guarantees [5]. Reddy and collaborators also studied monotonic and constrained interpolation by reducing fractal rational FIFs to classical interpolants under zero scaling factors [21]. In addition, Mahipal Reddy et al. visualized fractal evolution using MATLAB implementations [6]. Barnsley and his colleagues later described recurrent IFSs, elaborating connections with Markov chains, ergodic theory, Julia sets, and the collage theorem [10]. Yun and co-authors generated fractal surfaces using recurrent fractal curves via Lipschitz-based techniques [27]. Gowrisankar et al. devised an affine recurrent FIF ensuring bounded interpolation between piecewise linear segments [4]. Lastly, Reddy et al. proposed a univariate constrained recurrent rational fractal interpolation framework based on a recurrent iterated function system, which enhances the flexibility of cubic spline construction, establishes uniform error bounds and convergence results, and provides sufficient conditions to preserve shape constraints and boundedness between piecewise functions, straight lines, and rectangular regions, as illustrated through numerical examples [22].

This paper provides an RIFS-based method for creating a constrained recurrent cubic fractal framework for piecewise linear interpolation. This method is flexible, efficient, and guarantees convergence. A new group of recurrent rational fractal interpolation functions is carefully studied for its limiting behavior and convergence properties. Also, the framework is integrated with decision tree-based forecasting, which uses recurrent cubic fractal interpolation to ensure that the predicted outputs are smooth and structurally sound. The new method makes it possible to use it in more areas of science and engineering.

In this work, the overall structure is organized as follows: Section 3 introduces the essential terminology and fundamental concepts related to the Recurrent Iterated Function System (RIFS). Section 4 presents the construction of the proposed Constrained Recurrent Cubic Fractal Model (CR-CFM). Section 5 provides a detailed convergence analysis of the developed model. Section 6 discusses the constraints imposed on the vertical scaling factors and shape parameters of the constrained cubic recurrent rational fractal model. Section 7 demonstrates the predictive capability of the model through Decision Tree–based forecasting.

2. Recurrent Iterated Function System

In this section, we describe the formulation of the Recurrent Iterated Function System using the available data set. $\{(\mathbf{s}_\kappa,\ \mathbf{t}_\kappa) : \kappa \in M_M = \{1, 2, \ldots, M\}\}$, where the data set is defined over a strictly increasing sequence of abscissa. Indicate $\mathcal{L} = [\mathbf{s}_1,\ \mathbf{s}_M]$, $\mathcal{L}^*_\kappa = [\mathbf{s}_\kappa,\ \mathbf{s}_{\kappa+1}]$. For each $\mathcal{L}^*_\kappa$, let the data lie in the interval $\mathcal{A}_\kappa = [\mathbf{s}_{1(\kappa)},\ \mathbf{s}_{2(\kappa)}]$ in such a way $\mathbf{s}_{\kappa+1} - \mathbf{s}_\kappa < \mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)}$, where $\mathbf{s}_{1(\kappa)}, \mathbf{s}_{2(\kappa)} \in \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M\}$, $1(\kappa) < 2(\kappa)$ and $1(\kappa), 2(\kappa) \in M_M$. Let $\mathbb{E}_\kappa : \mathcal{A}_\kappa \to \mathcal{L}^*_\kappa,\ \kappa \in M_{M-1} = \{1, 2, \ldots, M-1\}$, be contractive mapping defined by $\mathbb{E}_\kappa(\mathbf{s}) = a_\kappa \mathbf{s} + b_\kappa$ such that

$$\mathbb{E}_\kappa(\mathbf{s}_{1(\kappa)}) = \mathbf{s}_\kappa,\ \mathbb{E}_\kappa(\mathbf{s}_{2(\kappa)}) = \mathbf{s}_{\kappa+1}, \tag{2.1}$$

where a_κ and b_κ are computed according to (2.1).
We affirm that the mapping $\mathbb{F}_\kappa : \mathcal{A}_\kappa \times \mathrm{R} \to \mathrm{R}, \kappa \in M_{M-1}$, is well-defined if it meets the following conditions:

$$\begin{aligned} &\mathbb{F}_\kappa(\mathbf{s}_{1(\kappa)},\ \mathbf{t}_{1(\kappa)}) = \mathbf{t}_\kappa,\ \mathbb{F}_\kappa(\mathbf{s}_{2(\kappa)},\ \mathbf{t}_{2(\kappa)}) = \mathbf{t}_{\kappa+1}, \\ &|\mathbb{F}_\kappa(\mathbf{s}, \mathbf{t}_1) - \mathbb{F}_\kappa(\mathbf{s}, \mathbf{t}_2)| \le |\beta_\kappa||\mathbf{t}_1 - \mathbf{t}_2|, \quad |\beta_\kappa| < 1. \end{aligned} \tag{2.2}$$

The parameters $\beta'_\kappa s$ are interpreted as vertical scaling factors, and their collection $\beta = (\beta_1, \beta_2, \ldots, \beta_{N-1})$ is termed the scale vector. For index $\kappa \in M_{M-1}$, a transformation $\gamma_\kappa : \mathcal{A}_\kappa \times \mathrm{R} \to \mathcal{L}^*_\kappa \times \mathrm{R}$ is defined by $\gamma_\kappa(\mathbf{s}, \mathbf{t}) = \{\mathbb{E}_\kappa(\mathbf{s}), \mathbb{F}_\kappa(\mathbf{s}, \mathbf{t})\}$, where $\mathbb{E}_\kappa$ governs the horizontal mapping and $\mathbb{F}_\kappa$ controls the vertical dynamics. The collection of these transformations acting on the corresponding domains $\mathcal{A}_\kappa \times \mathrm{R}$ forms a recurrent iterated function system, which provides the mathematical framework for constructing recurrent fractal interpolation functions.

3. Recurrent Fractal Interpolation Model

This section introduces the formulation of a recurrent fractal interpolation function framework and the construction of a recurrent cubic fractal interpolation function controlled by a single shape parameter. To this end, we introduce the function space $\mathcal{G} = \{\mathcal{F} \in C(\mathcal{L}) : \mathcal{F}(\mathbf{s}_1) = \mathbf{t}_1 \text{ and } \mathcal{F}(\mathbf{s}_M) = \mathbf{t}_M\}$, which comprises all continuous functions defined on $\mathcal{L}$ that exactly interpolate the prescribed endpoint data. Endowing this space with the uniform metric establishes a robust analytical setting for defining the recurrent fractal interpolation scheme and for examining its structural and functional properties.

$$D(\mathcal{F}_1, \mathcal{F}_2) = \max\{|\mathcal{F}_1(\mathbf{s}) - \mathcal{F}_2(\mathbf{s})| : \mathbf{s} \in \mathcal{L}\},\ \text{for } \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{G}.$$

We define an operator $\mathcal{T} : \mathcal{G} \to \mathcal{G}$ by the relation:

$$(\mathcal{T}\mathcal{F})(\mathbf{s}) := \mathbb{F}_\kappa(\mathbb{E}_\kappa^{-1}(\mathbf{s}),\ \mathcal{F} \circ \mathbb{E}_\kappa^{-1}(\mathbf{s})), \quad \kappa \in M_{M-1}.$$

As $\mathcal{T}$ is a contraction mapping defined on the complete metric space, $(\mathcal{G},\ D)$, it follows from the Banach fixed-point Theorem that there exists a unique function $\mathcal{F} \in \mathcal{G}$ satisfying:

$$\mathcal{T}\mathcal{F}(\mathbf{s}) = \mathcal{F}(\mathbf{s}), \quad \forall \mathbf{s} \in \mathcal{L}.$$

This function $\mathcal{F}$ fulfills the functional equation:

$$\mathcal{F}(\mathbf{s}) = \mathbb{F}_\kappa(\mathbb{E}_\kappa^{-1}(\mathbf{s}), \mathcal{F} \circ \mathbb{E}_\kappa^{-1}(\mathbf{s})), \quad \text{for } \mathbf{s} \in \mathcal{L}^*_\kappa.$$

The graphical representation of the continuous function $\mathcal{F}$, which exactly interpolates the prescribed data points $\mathcal{F}(\mathbf{s}_\kappa) = \mathbf{t}_\kappa$ for $\kappa \in \{1, 2, \ldots, M\}$, is identified as a recurrent fractal interpolation function associated with the given recurrent iterated function system (RIFS). The theorem presented below rigorously characterizes the analytical conditions under which such a recurrent fractal interpolant exists and is uniquely determined.
Theorem 1: Let$\{(\mathbf{s}_\kappa, \mathbf{t}_\kappa) : i = 1, 2, \ldots, N\}$ be the interpolation data, $\mathbb{E}_\kappa(x)$ is an affine function satisfy (2.1), and a continuous function $\mathbb{F}_\kappa(\mathbf{s}, \mathbf{t}) = \beta_\kappa y + q_\kappa(\mathbf{s})$ satisfy (2.2) for each $\mathcal{A}_\kappa = [\mathbf{s}_{1(\kappa)},\ \mathbf{s}_{2(\kappa)}] \in [\mathbf{s}_1,\ \mathbf{s}_N]$.

Suppose for some integer $j > 0, |\beta_\kappa| \leq z a_\kappa^j,\ 0 < z < 1$ and $q_\kappa \in \mathcal{C}^j[\mathbf{s}_{1(\kappa)}, \mathbf{s}_{2(\kappa)}],\ \kappa \in \mathcal{L}_\kappa$. Let

$$F_{i,\alpha}(\mathbf{s},\mathbf{t}) = \frac{\beta_\kappa y + q_\kappa^\alpha(\mathbf{s})}{a_\kappa^\alpha},\ \mathbf{t}_{1(\kappa),\alpha} = \frac{q_1^\alpha(\mathbf{s}_{1(\kappa)})}{a_1^\alpha - \beta_1},\ \mathbf{t}_{2(\kappa),\alpha} = \frac{q_{N-1}^\alpha(\mathbf{s}_{2(\kappa)})}{a_{M-1}^\alpha - \beta_{M-1}}$$

for $\alpha = 1, 2, \ldots, j$.

If $\mathbb{F}_{i-1,\alpha}(\mathbf{s}_{2(\kappa)}, \mathbf{t}_{2(\kappa),\alpha}) = \mathbb{F}_{i,\alpha}(\mathbf{s}_{1(\kappa)}, \mathbf{t}_{1(\kappa),\alpha})$ for $\kappa = 1, 2, \ldots, N-1$ and $\alpha = 1, 2, \ldots, j$, then $\gamma_\kappa(\mathbf{s},\mathbf{t}) = \{(E_\kappa(\mathbf{s}), \mathbb{F}_\kappa(\mathbf{s},\mathbf{t})) : \kappa = 1, 2, \ldots, N-1\}$ determines a RFIF $f \in \mathcal{C}^j[\mathbf{s}_{1(\kappa)}, \mathbf{s}_{2(\kappa)}]$ and f^α is the RFIF determined by $\{(\mathbb{E}_\kappa(\mathbf{s}),\ \mathbb{F}_\kappa(\mathbf{s},\mathbf{t})) : \kappa \in M_{M-1}\}$ for $\alpha = 1, 2, \ldots, j$.

4. Construction of Recurrent Cubic Fractal Model

Consider a Hermite interpolation data set $\{(\mathbf{s}_\kappa, \mathbf{t}_\kappa, d_\kappa) : \kappa \in M_M\}$. For each, $\kappa \in M_{M-1}$, we construct a mapping $\mathbb{F}_\kappa : \mathcal{A}_\kappa \times \mathrm{R} \to \mathrm{R}$ satisfying the interpolation conditions $\mathbb{F}_\kappa(\mathbf{s}_{1(\kappa)},\ \mathbf{t}_{1(\kappa)}) = \mathbf{t}_\kappa, \mathbb{F}_\kappa(\mathbf{s}_{2(\kappa)},\ \mathbf{t}_{2(\kappa)}) = \mathbf{t}_{\kappa+1}$. The collection of maps $\gamma_\kappa(\mathbf{s},\mathbf{t}) = (\mathbb{E}_\kappa(\mathbf{s}),\ \mathbb{F}_\kappa(\mathbf{s},\mathbf{t}))$ then forms a recurrent iterated function system, where d_κ denotes the proposed the prescribed derivative at β_κ represents the corresponding vertical scaling factor. Here, the horizontal and vertical components are chosen as an affine map $\mathbb{E}_\kappa(\mathbf{s}) = a_\kappa \mathbf{s} + b_\kappa$ and a rational function $\mathbb{F}_\kappa(\mathbf{s},\mathbf{t}) = \beta_\kappa \mathbf{t} + \frac{P_\kappa(\mathbf{s})}{Q_\kappa(\mathbf{s})}$, respectively. As a result, the associated rational recurrent fractal interpolation function Φ satisfies the following functional equation.

$$\Phi(E_\kappa(\mathbf{s})) = \beta_\kappa \Phi(\mathbf{s}) + \frac{P_\kappa(\mathbf{s})}{Q_\kappa(\mathbf{s})},\ \mathbf{s} \in \mathcal{A}_\kappa, \tag{4.1}$$

$$\begin{aligned} \text{where}\ \ &\mathbb{P}_\kappa(\mathbf{s}) = \mathbb{T}_{1,i}(1-\delta)^3 + \mathbb{T}_{2,i}\delta(1-\delta)^2 + \mathbb{T}_{3,i}\delta^2(1-\delta) + \mathbb{T}_{4,i}\delta^3, \\ &\mathbb{Q}_\kappa(x) = 1 + (r_\kappa - 2)\delta(1-\delta), \end{aligned}$$

$\mathbb{T}_{1,i}, \mathbb{T}_{2,i}, \mathbb{T}_{3,i}, \mathbb{T}_{4,i}$, are the coefficient of $\mathbb{P}_\kappa(\mathbf{s}),\ r_\kappa$ is the shape parameter $\delta = \frac{\mathbf{s}-\mathbf{s}_{1(\kappa)}}{x_{2(\kappa)}-x_{1(\kappa)}},\ \mathbf{s} \in \mathcal{A}_\kappa$.
Put $\mathbf{s} = \mathbf{s}_{1(\kappa)},\ \delta = 0$ and $\mathbf{s} = \mathbf{s}_{2(\kappa)},\ \delta = 1$ in (4.1), we get

$$\mathbb{P}_\kappa(\mathbf{s}_{1(\kappa)}) = \mathbb{T}_{1,i},\quad \mathbb{P}_\kappa(\mathbf{s}_{1(\kappa)}) = 1 \text{ and } \mathbb{P}_\kappa(\mathbf{s}_{2(\kappa)}) = \mathbb{T}_{4,i},\quad \mathbb{P}_\kappa(\mathbf{s}_{2(\kappa)}) = 1.$$

Substitute $\mathbf{s} = \mathbf{s}_{1(\kappa)}$ and $\mathbf{s} = \mathbf{s}_{2(\kappa)}$ in (4.1), we obtain $\mathbb{T}_{1,i} = \mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)}$
and $\mathbb{T}_{4,i} = \mathbf{t}_{\kappa+1} - \beta_\kappa \mathbf{t}_{2(\kappa)}$.
Derivative of Φ in (4.1) with respect to $\mathbf{s}$ is

$$\Phi'(\mathbb{E}_\kappa(\mathbf{s}))a_\kappa = \beta_\kappa \Phi'(\mathbf{s}) + \frac{\mathbb{Q}_\kappa(\mathbf{s})\mathbb{P}'_\kappa(\mathbf{s}) - \mathbb{P}_\kappa(\mathbf{s})\mathbb{Q}'_\kappa(\mathbf{s})}{(\mathbb{Q}_\kappa(\mathbf{s}))^2}. \tag{4.2}$$

Substitute, $\mathbb{E}_\kappa(\mathbf{s}_{1(\kappa)}) = \mathbf{s}_\kappa,\ \mathbb{E}_\kappa(\mathbf{s}_{2(\kappa)}) = \mathbf{s}_{\kappa+1}$ and $\mathbb{T}_{1,i}, \mathbb{T}_{4,i}$ expression in (4.2), we get

$$\begin{aligned} \mathbb{T}_{2,i} &= (r_\kappa + 1)(\mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)}) + h_\kappa d_\kappa - \beta_\kappa(d_{1(\kappa)}(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})), \\ \mathbb{T}_{3,i} &= (r_\kappa + 1)(\mathbf{t}_{\kappa+1} - \beta_\kappa \mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1} + \beta_\kappa(d_{2(\kappa)}(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})). \end{aligned}$$

We obtained a recurrent cubic fractal model as

$$\begin{aligned} &\Phi(E_\kappa(\mathbf{s})) = \beta_\kappa \Phi(\mathbf{s}) + \frac{P_\kappa(\mathbf{s})}{Q_\kappa(\mathbf{s})}, \\ &P_\kappa(\mathbf{s}) = (\mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)})(1-\delta)^3 + \{(r_\kappa + 1)(\mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)}) + h_\kappa d_\kappa - \beta_\kappa(d_{1(\kappa)}(\mathbf{s}_{2(\kappa)} \\ &\qquad - \mathbf{s}_{1(\kappa)}))\}\delta(1-\delta)^2 + \{(r_\kappa + 1)(\mathbf{t}_{\kappa+1} - \beta_\kappa \mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1} + \beta_\kappa(d_{2(\kappa)}(\mathbf{s}_{2(\kappa)} \\ &\qquad - \mathbf{s}_{1(\kappa)}))\}\delta^2(1-\delta) + (\mathbf{t}_{\kappa+1} - \beta_\kappa \mathbf{t}_{2(\kappa)})\delta^3, \\ &Q_\kappa(\mathbf{s}) = 1 + (r_\kappa - 2)\delta(1-\delta),\quad \delta = \frac{\mathbf{s} - \mathbf{s}_{1(\kappa)}}{\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)}},\quad \mathbf{s} \in \mathcal{A}_\kappa. \end{aligned}$$

5. Convergence Analysis

We now analyze the convergence behavior. Assume that the Hermite data $\{(\mathbf{s}_\kappa, \mathbf{t}_\kappa) : \kappa \in M_M\}$ are generated by a smooth function $\Psi \in C^1(\mathcal{L})$. Let C be the associated classical interpolation of the RRFIF. The convergence of the proposal model Φ towards Ψ is achieved by first proving that C converges to Ψ and subsequently estimating the uniform distance between Φ and C. Using the triangle inequality, we obtain

$$\|\Phi - \Psi\|_\infty \leq \|\Phi - C\|_\infty + \|C - \Psi\|_\infty.$$

We have $\Phi(E_\kappa(\mathbf{s})) = \beta_\kappa \Phi(\mathbf{s}) + \frac{P_\kappa(\mathbf{s})}{Q_\kappa(\mathbf{s})}$. If $\beta_\kappa = 0$ then, we get

$$\begin{aligned}
C(x) &= \frac{\mathbb{P}_\kappa(\mathbf{s})}{\mathbb{Q}_\kappa(\mathbf{s})} = \frac{\mathbb{T}_{1,i}(1-\lambda)^3 + \mathbb{T}_{2,i}\lambda(1-\lambda)^2 + \mathbb{T}_{3,i}\lambda^2(1-\lambda) + \mathbb{T}_{4,i}\lambda^3}{1 + (r_\kappa - 2)\lambda(1-\lambda)},\\
\mathbb{P}_\kappa(\mathbf{s}) &= \mathbf{t}_\kappa(1-\lambda)^3 + \{(r_\kappa+1)\mathbf{t}_\kappa + h_\kappa d_\kappa\}\lambda(1-\lambda)^2 + \{(r_\kappa+1)\mathbf{t}_{\kappa+1} - h_\kappa d_{\kappa+1}\}\lambda^2(1-\lambda)\\
&\quad + \mathbf{t}_{\kappa+1}\lambda^3,\\
\mathbb{Q}_\kappa(\mathbf{s}) &= 1 + (r_\kappa - 2)\lambda(1-\lambda),\ \lambda = \frac{\mathbf{s} - \mathbf{s}_\kappa}{\mathbf{s}_{\kappa+1} - \mathbf{s}_\kappa},\ \mathbf{s} \in [\mathbf{s}_\kappa,\ \mathbf{s}_{\kappa+1}],\ \kappa \in M_{M-1}.
\end{aligned}$$

Now, we express the difference between $\Phi(x)$ and $C(x)$ as:

$$\begin{aligned}
\Phi(\mathbf{s}) - C(\mathbf{s}) = \frac{1}{\mathbb{Q}_\kappa(\lambda)}\Big\{ -\beta_\kappa\big[\mathbf{t}_{1(\kappa)}(1-\lambda)^3 + \mathbf{t}_{2(\kappa)}\lambda^3 + \big(r_\kappa+1)\mathbf{t}_{1(\kappa)} - d_{1(\kappa)}(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\big)\\
\lambda(1-\lambda)^2\big] + \big[(r_\kappa+1)y_{2(\kappa)} + d_{2(\kappa)}(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\big]\lambda^2(1-\lambda)\Big\}.
\end{aligned}$$

We can rewrite it as:

$$\Phi(\mathbf{s}) - C(\mathbf{s}) = \frac{1}{\mathbb{Q}_\kappa(\lambda)}\left\{-\beta_\kappa[V_0\mathbf{t}_{1(\kappa)} + V_1\mathbf{t}_{2(\kappa)} + V_2 d_{1(\kappa)} + V_3 d_{2(\kappa)}]\right\},$$

$$\begin{aligned}
&\text{where } V_0 = (1-\lambda)^3 + (r_\kappa+1)\lambda(1-\lambda)^2,\ V_1 = (r_\kappa+1)\lambda^2(1-\lambda) + \lambda^3,\\
&\qquad V_2 = h_\kappa\lambda(1-\lambda)^2,\ V_3 = -h_\kappa\lambda^2(1-\lambda).\\
&\qquad\qquad \frac{V_0+V_1}{\mathbb{Q}_\kappa(\lambda)} = 1, \quad \frac{V_2+V_3}{\mathbb{Q}_\kappa(\lambda)} = \frac{h_\kappa\lambda(1-\lambda)(1-2\lambda)}{1+(r_\kappa-2)\lambda(1-\lambda)},\\
&\Phi(\mathbf{s}) - C(\mathbf{s}) = \frac{-\beta_\kappa}{\mathbb{Q}_\kappa(\lambda)}\left[V_0\mathbf{t}_{1(\kappa)} + V_1\mathbf{t}_{2(\kappa)} + V_2 d_{1(\kappa)} + V_3 d_{2(\kappa)}\right].\\
&|\Phi(\mathbf{s}) - C(\mathbf{s})| \leq |\beta_\kappa|\left[\max\{\mathbf{t}_{1(\kappa)}, \mathbf{t}_{2(\kappa)}\} + V(\lambda,h)\max\{d_{1(\kappa)}, d_{2(\kappa)}\}\right],\\
&\text{where}\quad V(\lambda,h) = \frac{h_\kappa\lambda(1-\lambda)(1-2\lambda)}{1+(r_\kappa-2)\lambda(1-\lambda)}.\\
&\|\Phi(\mathbf{s}) - C(\mathbf{s})\|_\infty \leq |\beta|_\infty|y|_\infty + V(\Phi;h)|d|_\infty.\\
&\text{Now, } \Psi(\mathbf{s}) - C(\mathbf{s}) = (1-\lambda)^3[\Psi(\mathbf{s}) - y_\kappa] + (r_\kappa+1)\lambda(1-\lambda)^2[\Psi(\mathbf{s}) - \mathbf{t}_\kappa]\\
&\qquad + (r_\kappa+1)\lambda^2(1-\lambda)[\Psi(\mathbf{s}) - \mathbf{t}_{\kappa+1}] + \lambda^3[\Psi(\mathbf{s}) - \mathbf{t}_{\kappa+1}]\\
&\qquad - \lambda(1-\lambda)^2 d_\kappa + d_{\kappa+1}\lambda^2(1-\lambda).
\end{aligned}$$

The maximum error can be evaluated using the infinity norm:

$$\begin{aligned}
&\Psi(\mathbf{s}) - C(\mathbf{s}) = \frac{1}{\mathbb{Q}_\kappa(\mathbf{s})}\{(\Psi(\mathbf{s}) - \mathbf{t}_\kappa)W_0 + (\Psi(\mathbf{s}) - \mathbf{t}_{\kappa+1})W_1 + W_2 d_\kappa + W_3 d_{\kappa+1}]\}.\\
&\text{Where } W_0 = (1-\lambda)^3 + (r_\kappa+1)\lambda(1-\lambda)^2,\ W_1 = (r_\kappa+1)\lambda^2(1-\lambda) + \lambda^3,\\
&\qquad W_2 = -h_\kappa\lambda(1-\lambda)^2,\ W_3 = h_\kappa\lambda^2(1-\lambda).\\
&\text{Now } \frac{W_0+W_1}{\mathbb{Q}_\kappa(\mathbf{s})} = 1,\ \frac{W_2+W_3}{\mathbb{Q}_\kappa(\mathbf{s})} = \frac{-h_\kappa\lambda(1-\lambda)(1-2\lambda)}{\mathbb{Q}_\kappa(\mathbf{s})}.
\end{aligned}$$

$$|\Psi(\mathbf{s})-C(\mathbf{s})|\leq\frac{1}{Q_\kappa(\lambda_j)}\left[\left(\frac{1}{4}|r_\kappa|+1\right)W(\Psi;h)+\frac{1}{4}\max\{|d_\kappa|,|d_{\kappa+1}|\}\right],$$

$$\text{where}\quad W(\Psi;h)=\max\{|\Psi(\mathbf{s})-\Psi^*(\mathbf{s})|:|\mathbf{s}-\mathbf{s}^*|<h,\mathbf{s},\mathbf{s}^*\in\mathcal{L}\}.$$

We can express it as:

$$\|\Psi(\mathbf{s})-C(\mathbf{s})\|_\infty\leq|\beta|_\infty|\mathbf{t}|_\infty+W(\Psi,h)|d|_\infty,$$

where,

$$\begin{aligned}
|\mathbf{t}|_\infty&=\max\{|\mathbf{t}_{1(\kappa)}|,|\mathbf{t}_{2(\kappa)}|:i=1,2,\ldots,N\}\leq\max\{|\mathbf{t}_\kappa|:i=1,2,\ldots,N\},\\
|d|_\infty&=\max\{|d_{1(\kappa)}|,|d_{2(\kappa)}|:i=1,2,\ldots,N\}\leq\max\{|d_\kappa|_\infty:i=1,2,\ldots,N\},\\
|h|_\infty&=\max\{|h_\kappa|:i=1,2,\ldots,N-1\},\quad |r|_\infty=\max\{|r_\kappa|:i=1,2,\ldots,N-1\}.
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } |\Phi(\mathbf{s})-\Psi(\mathbf{s})|\leq{}&|\beta_\kappa|\left[\max\{\mathbf{t}_{1(\kappa)},\mathbf{t}_{2(\kappa)}\}+V(\lambda,h)\max\{d_{1(\kappa)},d_{2(\kappa)}\}\right]\\
&+\frac{1}{Q_\kappa(\lambda_j)}\left[\left(\frac{1}{4}|r_\kappa|+1\right)W(\Psi;h)+\frac{1}{4}\max\{|d_\kappa|,|d_{\kappa+1}|\}\right].
\end{aligned}$$

Convergence Result: Let Φ be uniformly continuous on $\mathcal{L}$. Then its modulus of continuity $W(\Psi;h)\to 0$, vanishes as the mesh size $h\to 0$. Consequently, by comparing Ψ with its classical analogue C, one can rigorously derive an explicit upper estimate for the associated approximation error, thereby establishing the convergence of the proposed scheme.

$$\|\Phi-\Psi\|_\infty\leq\|\Phi-C\|_\infty+\|C-\Psi\|_\infty.$$

6. Constraints of the Recurrent Cubic Fractal Model

This section delineates the form parameters and vector scaling factor that satisfy the requirements of the recurrent cubic fractal model. Our primary aim is to develop a restricted RCFM such that the interpolation dataset, $\{(\mathbf{s}_\kappa,\ \mathbf{t}_\kappa):\kappa\in M_M\}$, is confined to the boundaries of two piecewise linear functions: $\mathcal{G}^l_\kappa=\Omega_\kappa\mathbf{s}+\mu_\kappa$ and $\mathcal{G}^u_\kappa=\Omega^*_\kappa\mathbf{s}+\mu^*_\kappa$. Considering that RCFM is generated by an implicit method, we will analyze two cases: $\beta_\kappa\geq 0$ and $\beta_\kappa<0$ for all $\kappa\in M_{M-1}$. The subsequent condition must be verified to ensure that the assertions provided by the iterative procedure adhere to these regulations when piecewise linear functions encompass the dataset:

$$\begin{aligned}
&\mathcal{G}^l_\kappa(\mathbb{E}_\kappa(\mathbf{s}))\leq f(\mathbb{E}_\kappa(\mathbf{s}))\leq\mathcal{G}^u_\kappa(\mathbb{E}_\kappa(x)),\ \mathbb{E}(\mathbf{s}_j)=a_\kappa\mathbf{s}_j+b_\kappa,\ j\in M_M,\\
&\Rightarrow\Omega_\kappa(\mathbb{E}_\kappa(\mathbf{s}_j))+\mu_\kappa\leq\beta_\kappa f(\mathbf{s}_j)+\frac{P_\kappa(\sigma_j)}{Q_\kappa(\sigma_j)}\leq\Omega^*_\kappa(\mathbb{E}_\kappa(\mathbf{s}_j))+\mu^*_\kappa,\ \sigma_j=\frac{\mathbf{s}_j-\mathbf{s}_{1(\kappa)}}{\mathbf{s}_{2(\kappa)}-\mathbf{s}_{1(\kappa)}}.\\
&\text{Here }\ f(\mathbf{s}_j)\geq\Omega_\kappa x_j+\mu_\kappa\Rightarrow\beta_\kappa f(\mathbf{s}_j)\geq\beta_\kappa(\Omega_\kappa\mathbf{s}_j+\mu_\kappa),\\
&\quad\Rightarrow\beta_\kappa f(\mathbf{s}_j)Q_\kappa(\sigma_j)\geq\beta_\kappa(\Omega_\kappa\mathbf{s}_j+\mu_\kappa)Q_\kappa(\sigma_j).
\end{aligned}$$

Our aim is now to derive suitable conditions on the RIFS parameters that ensure the satisfaction of the following inequalities:

$$\Omega_\kappa(a_\kappa\mathbf{s}+b_\kappa)+\mu_\kappa\leq\beta_\kappa f(\mathbf{s})+\frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)}\quad\text{and}\quad\beta_\kappa f(\mathbf{s})+\frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)}\leq\Omega^*_\kappa(a_\kappa\mathbf{s}+b_\kappa)+\mu^*_\kappa.\tag{6.1}$$

Theorem 2. Let the set of points $\{(\mathbf{s}_\kappa,\ \mathbf{t}_\kappa,\ d_\kappa):\kappa\in M_M\}$ represent Hermite data located above the given piecewise linear functions. If the lower and upper bounding functions are defined as $\mathcal{G}^l_\kappa=\Omega_\kappa\mathbf{s}+\mu_\kappa$ and $\mathcal{G}^u_\kappa=\Omega^*_\kappa\mathbf{s}+\mu^*_\kappa,\ \kappa\in M_{M-1}$, then the constructed RCFM will lie entirely above these bounding functions, provided that the recurrent IFS parameters are chosen suitably.

$$(i)\ 0 \leq \beta_\kappa < \min\left[a_\kappa,\ \frac{\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)}{\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)})},\ \frac{\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})}{\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})},\right.$$
$$\left.\frac{\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa}{\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}},\ \frac{\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{i+1}}{\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}}\right],$$

$$(ii)\ r_\kappa > \max\left[0,\ \frac{\mathbf{N}_1}{\mathbf{N}_2},\ \frac{\mathbf{N}_3}{\mathbf{N}_4},\ \frac{\mathbf{N}_5}{\mathbf{N}_6},\ \frac{\mathbf{N}_7}{\mathbf{N}_8}\right],$$

where

$$\begin{aligned}
\mathbf{N}_1 &= -[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)})) + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + \Omega_\kappa(\beta_\kappa - a_\kappa)|J|],\\
\mathbf{N}_2 &= (\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}))],\\
\mathbf{N}_3 &= -[(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|],\\
\mathbf{N}_4 &= (\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1}))) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})) + \Omega_\kappa(\beta_\kappa - a_\kappa)|J|,\\
\mathbf{N}_5 &= -[(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}) - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + \Omega^*_\kappa(a_\kappa - \beta_\kappa)|J|],\\
\mathbf{N}_6 &= (\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}),\\
\mathbf{N}_7 &= -[(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)})] - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|,\\
\mathbf{N}_8 &= (\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) + \Omega^*_\kappa(a_\kappa - \beta_\kappa)|J|.
\end{aligned}$$

proof. The left-hand side inequality of (6.1) can be expressed as,

$$\beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) - \{[\Omega_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j)\} \geq 0.$$

Assume that $\beta_\kappa \geq 0$.Thus, $f(\mathbf{s}) \geq \Omega_\kappa\mathbf{s} + \mu_\kappa \Rightarrow \beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) \geq \beta_\kappa(\Omega_\kappa\mathbf{s} + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j)$.

Consequently, the required criteria inherently ensure the validity of the following inequalities.

$$\begin{aligned}
&\beta_\kappa(\Omega_\kappa\mathbf{s} + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) - (\Omega_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j) \geq 0,\\
&\Rightarrow \beta_\kappa[\Omega_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)}) + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) - [\Omega_\kappa(a_\kappa[\mathbf{s}_{1(\kappa)} + \sigma_j(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)}) + b_\kappa]\\
&\quad + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) \geq 0,\\
&\Rightarrow \beta_\kappa[\Omega_\kappa\mathbf{s}_{1(\kappa)} + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) + \beta_\kappa\Omega_\kappa(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\sigma_j\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) - [\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)} + b_\kappa) + \mu_\kappa]\\
&\qquad \mathbb{Q}_\kappa(\sigma_j) - \Omega_\kappa a_\kappa(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\sigma_j\mathbb{Q}_\kappa(\sigma_j) \geq 0.
\end{aligned}$$

Which can be rewritten as,

$$\begin{aligned}
&\beta_\kappa[\Omega_\kappa\mathbf{s}_{1(\kappa)} + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) + \beta_\kappa(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\sigma_j\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j)\\
&- [\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)} + b_\kappa) + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) - \Omega_\kappa a_\kappa(\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)})\sigma_j\mathbb{Q}_\kappa(\sigma_j) \geq 0.
\end{aligned} \tag{6.2}$$

Using degree elevation technique on $\mathbb{Q}_\kappa(\sigma_j)$ and $\sigma_j\mathbb{Q}_\kappa(\sigma_j)$, we obtain,

$$\begin{aligned}
\mathbb{Q}_\kappa(\sigma_j) &= (1-\sigma_j)^3 + (r_\kappa + 1)(1-\sigma_j)^2\sigma_j + (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3,\\
\sigma_j\mathbb{Q}_\kappa(\sigma_j) &= (1-\sigma_j)^2\sigma_j + r_\kappa\sigma_j^2(1-\sigma_j) + \sigma_j^3.
\end{aligned} \tag{6.3}$$

Using (6.2)-(6.3), we get

$$\begin{aligned}
&\{\beta_\kappa(\Omega_\kappa\mathbf{s}_{1(\kappa)} + \mu_\kappa) - [\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)} + b_\kappa) + \mu_\kappa]\}\{(1-\sigma_j)^3 + (r_\kappa + 1)(1-\sigma_j)^2\sigma_j\\
&+ (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3\} + [\Omega_\kappa(\beta_\kappa - a_\kappa)|J|]\{(1-\sigma_j)^2\sigma_j + r_\kappa\sigma_j^2(1-\sigma_j) + \sigma_j^3\}\\
&+ \{[(\mathbf{t}_\kappa - \beta_\kappa\mathbf{t}_{1(\kappa)})](1-\sigma_j)^3 + [(r_\kappa + 1)(\mathbf{t}_\kappa - \beta_\kappa\mathbf{t}_{1(\kappa)}) + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J|](1-\sigma_j)^2\sigma_j\\
&+ [(r_\kappa + 1)(\mathbf{t}_{i+1} - \beta_\kappa\mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|](1-\sigma_j)\sigma_j^2 + [(\mathbf{t}_{i+1} - \beta_\kappa\mathbf{t}_{2(\kappa)})]\sigma_j^3\} \geq 0.
\end{aligned}$$

By rearranging and simplifying the given inequality, we obtain

$$\begin{aligned}\Rightarrow &[(\mathbf{t}_\kappa-(\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu_\kappa))-\beta_\kappa(\mathbf{t}_{1(\kappa)}-(\Omega_\kappa\mathbf{s}_{1(\kappa)}+\mu_\kappa))](1-\sigma_j)^3+\{(r_\kappa+1)\\ &[(\mathbf{t}_\kappa-(\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu_\kappa))-\beta_\kappa(\mathbf{t}_{1(\kappa)}-(\Omega_\kappa\mathbf{s}_{1(\kappa)}+\mu_\kappa))]+[h_\kappa d_\kappa-\beta_\kappa d_{1(\kappa)}|J|\\ &+\Omega_\kappa(\beta_\kappa-a_\kappa)|J|]\}(1-\sigma_j)^2\sigma_j+\{(r_\kappa+1)[(\mathbf{t}_{i+1}-(\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu_\kappa))-\beta_\kappa(\mathbf{t}_{2(\kappa)}\\ &-(\Omega_\kappa\mathbf{s}_{1(\kappa)}+\mu_\kappa))]-h_\kappa d_{\kappa+1}+\beta_\kappa d_{2(\kappa)}|J|+r_\kappa(\Omega_\kappa(\beta_\kappa-a_\kappa)|J|)\}\sigma_j^2(1-\sigma_j)+\{[(\mathbf{t}_{i+1}\\ &-(\Omega_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu_\kappa))-\beta_\kappa(\mathbf{t}_{2(\kappa)}-(\Omega_\kappa\mathbf{s}_{1(\kappa)}+\mu_\kappa))+(\Omega_\kappa(\beta_\kappa-a_\kappa)|J|)\}\sigma_j^3\geq 0.\end{aligned}$$

Therefore, the inequality stated above holds true under the following conditions:

(i) $0\leq\beta_\kappa<\min\left[a_\kappa,\ \dfrac{\mathbf{t}_\kappa-\mathcal{G}_\kappa^l(\mathbf{s}_\kappa)}{\mathbf{t}_{1(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{1(\kappa)})},\ \dfrac{\mathbf{t}_{i+1}-\mathcal{G}_\kappa^l(\mathbf{s}_{\kappa+1})}{\mathbf{t}_{2(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{2(\kappa)})}\right]$

(ii) $r_\kappa>\max\left[0,\ \dfrac{\mathbf{N}_1}{\mathbf{N}_2},\ \dfrac{\mathbf{N}_3}{\mathbf{N}_4}\right]$,

where,

$\mathbf{N}_1=-[(\mathbf{t}_\kappa-\mathcal{G}_\kappa^l(\mathbf{s}_\kappa))-\beta_\kappa(\mathbf{t}_{1(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{1(\kappa)}))+h_\kappa d_\kappa-\beta_\kappa d_{1(\kappa)}|J|+\Omega_\kappa(\beta_\kappa-a_\kappa)|J|]$,

$\mathbf{N}_2=(\mathbf{t}_\kappa-\mathcal{G}_\kappa^l(\mathbf{s}_\kappa))-\beta_\kappa(\mathbf{t}_{1(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{1(\kappa)}))]$,

$\mathbf{N}_3=-[(\mathbf{t}_{i+1}-\mathcal{G}_\kappa^l(\mathbf{s}_{\kappa+1}))-\beta_\kappa(\mathbf{t}_{2(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{2(\kappa)}))-h_\kappa d_{\kappa+1}+\beta_\kappa d_{2(\kappa)}|J|]$,

$\mathbf{N}_4=(\mathbf{t}_{i+1}-\mathcal{G}_\kappa^l(\mathbf{s}_{\kappa+1})))-\beta_\kappa(\mathbf{t}_{2(\kappa)}-\mathcal{G}_\kappa^l(\mathbf{s}_{2(\kappa)}))+\Omega_\kappa(\beta_\kappa-a_\kappa)|J|$.

The right-hand side inequality of (6.1) can be expressed as,

$$\mathcal{G}_\kappa^l(\mathbb{E}_\kappa(x))\leq f(\mathbb{E}_\kappa(x))\leq\mathcal{G}_\kappa^u(\mathbb{E}_\kappa(x)),\ \mathbb{E}(x_j)=a_\kappa x_j+b_\kappa,\ j\in M_M.$$

$$\Omega_\kappa(\mathbb{E}_\kappa(\mathbf{s}))+\mu_\kappa\leq\beta_\kappa f(\mathbf{s}_\kappa)+\frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)}\leq\Omega_\kappa^*(\mathbb{E}_\kappa(\mathbf{s}))+\mu_\kappa^*,\quad \sigma_j=\frac{\mathbf{s}-\mathbf{s}_{1(\kappa)}}{\mathbf{s}_{2(\kappa)}-\mathbf{s}_{1(\kappa)}},\ j\in M_M.$$

Right inequality of the above expression is considered.

$$\Rightarrow\beta_\kappa f(\mathbf{s})+\frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)}\leq\Omega_\kappa^*(a_\kappa\mathbf{s}+b_\kappa)+\mu_\kappa^*.$$

Multiplying both sides of the inequality by the positive quantity $\mathbb{Q}_\kappa(\sigma_j)$, we obtain

$$\Rightarrow\beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j)+\mathbb{P}_\kappa(\sigma_j)-\mathbb{Q}_\kappa(\sigma_j)[\Omega_\kappa^*(a_\kappa\mathbf{s}+b_\kappa)+\mu_\kappa^*]\leq 0.$$

Using the assumed lower bound on the function f, namely $f(\mathbf{s})\geq\Omega_\kappa^*\mathbf{s}+\mu_\kappa^*$, and since $\beta_\kappa>0$, we have

$$\begin{aligned}\text{Here } f(\mathbf{s})\geq\Omega_\kappa^*\mathbf{s}+\mu_\kappa^*&\Rightarrow\beta_\kappa f(\mathbf{s})\geq\beta_\kappa(\Omega_\kappa^*\mathbf{s}+\mu_\kappa^*)\\ &\Rightarrow\beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j)\geq\beta_\kappa(\Omega_\kappa^*\mathbf{s}+\mu_\kappa^*)\mathbb{Q}_\kappa(\sigma_j).\end{aligned}$$

Substituting this estimate into the previous inequality yields

$$\Rightarrow\beta_\kappa(\Omega_\kappa^*\mathbf{s}+\mu_\kappa^*)\mathbb{Q}_\kappa(\sigma_j)+\mathbb{P}_\kappa(\sigma_j)-(\mathbb{Q}_\kappa(\sigma_j))[\Omega_\kappa^*(a_\kappa\mathbf{s}+b_\kappa)+\mu_\kappa^*]\leq 0.$$

Next, expressing $\mathbf{s}$ in terms of the local coordinate $\mathbf{s}=\mathbf{s}_{1(\kappa)}+\sigma_j|J|$, the inequality can be rewritten as

$$\begin{aligned}\Rightarrow&[\beta_\kappa(\Omega_\kappa^*(\mathbf{s}_{1(\kappa)}+\mu_\kappa^*)-\Omega_\kappa^*(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu_\kappa^*)]\mathbb{Q}_\kappa(\sigma_j)\\ &-[\beta_\kappa\Omega_\kappa^*|J|-\Omega_\kappa^*a_\kappa|J|]\sigma_j\mathbb{Q}_\kappa(\sigma_j)+\mathbb{P}_\kappa(\sigma_j)\leq 0.\end{aligned}$$

By performing degree elevation on $\mathbb{Q}_\kappa(\sigma_j)$ and $\sigma_j\mathbb{Q}_\kappa(\sigma_j)$, we derive

$$\begin{aligned}&[(\Omega^*_\kappa(a_\kappa \mathbf{s}_{1(\kappa)}+b_\kappa)+\mu^*_\kappa)-\beta_\kappa(\Omega^*_\kappa\mathbf{s}_{1(\kappa)}+\mu^*_\kappa)]\{((1-\sigma_j)^3+(r_\kappa+1)(1-\sigma_j)^2\sigma_j\\&+(r_\kappa+1)\sigma_j^2(1-\sigma_j)+\sigma_j^3\}+[\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|]\{(1-\sigma_j)^2\sigma_j+r_\kappa\sigma_j^2(1-\sigma_j)+\sigma_j^3\}\\&-\{[(\mathbf{t}_\kappa-\beta_\kappa\mathbf{t}_{1(\kappa)})](1-\sigma_j)^3+[(r_\kappa+1)(\mathbf{t}_\kappa-\beta_\kappa\mathbf{t}_{1(\kappa)})+h_\kappa d_\kappa-\beta_\kappa d_{1(\kappa)}|J|](1-\sigma_j)^2\sigma_j\\&+[(r_\kappa+1)(\mathbf{t}_{\kappa+1}-\beta_\kappa\mathbf{t}_{2(\kappa)})-h_\kappa d_{\kappa+1}+\beta_\kappa d_{2(\kappa)}|J|](1-\sigma_j)\sigma_j^2+[(\mathbf{t}_{i+1}-\beta_\kappa\mathbf{t}_{2(\kappa)})]\sigma_j^3\}\geq 0.\end{aligned}$$

Rewriting and simplifying the inequality, we derive

$$\begin{aligned}\Rightarrow\ &(r_\kappa-1)[((\Omega^*_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu^*_\kappa)-\mathbf{t}_\kappa)-\beta_\kappa((\Omega^*_\kappa\mathbf{s}_{1(\kappa)}+\mu^*_\kappa)-\mathbf{t}_{1(\kappa)}](1-\sigma_j)^3\\&+\{(2r_\kappa-1)[((\Omega^*_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu^*_\kappa)-\mathbf{t}_\kappa)-\beta_\kappa((\Omega^*_\kappa\mathbf{s}_{1(\kappa)}+\mu^*_\kappa)-\mathbf{t}_{1(\kappa)}]\\&+(r_\kappa-1)[\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|-h_\kappa d_\kappa+\beta_\kappa d_{1(\kappa)}|J|]\}(1-\sigma_j)^2\sigma_j\\&+\{(r_\kappa+1)[((\Omega^*_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu^*_\kappa)-\mathbf{t}_{\kappa+1})-\beta_\kappa((\Omega^*_\kappa\mathbf{s}_{1(\kappa)}+\mu^*_\kappa)-\mathbf{t}_{2(\kappa)})]\\&+r_\kappa[\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|]+h_\kappa d_{\kappa+1}-\beta_\kappa d_{2(\kappa)}|J|\}\sigma_j^2(1-\sigma_j)+[((\Omega^*_\kappa(a_\kappa\mathbf{s}_{1(\kappa)}+b_\kappa)+\mu^*_\kappa)\\&-\mathbf{t}_{\kappa+1})-\beta_\kappa((\Omega^*_\kappa\mathbf{s}_{1(\kappa)}+\mu^*_\kappa)-\mathbf{t}_{2(\kappa)})+\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|]\sigma_j^3\geq 0.\end{aligned}$$

Therefore, the aforementioned inequality is applicable when the subsequent criteria are satisfied.

(iii) $0\leq\beta_\kappa<\min\left[a_\kappa,\ \dfrac{\mathcal{G}^u_\kappa(\mathbf{s}_\kappa)-\mathbf{t}_\kappa}{\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})-\mathbf{t}_{1(\kappa)}},\ \dfrac{\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1})-\mathbf{t}_{\kappa+1}}{\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})-\mathbf{t}_{2(\kappa)}}\right]$,

(iv) $r_\kappa>\max\left[0,\ \dfrac{\mathbf{N}_5}{\mathbf{N}_6},\ \dfrac{\mathbf{N}_7}{\mathbf{N}_8}\right]$,

where,

$\mathbf{N}_5=-[(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa)-\mathbf{t}_\kappa)-\beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})-\mathbf{t}_{1(\kappa)})-h_\kappa d_\kappa+\beta_\kappa d_{1(\kappa)}|J|+\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|]$,

$\mathbf{N}_6=(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa)-\mathbf{t}_\kappa)-\beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})-\mathbf{t}_{1(\kappa)})$,

$\mathbf{N}_7=-[(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1})-\mathbf{t}_{\kappa+1})-\beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})-\mathbf{t}_{2(\kappa)})]-h_\kappa d_{\kappa+1}+\beta_\kappa d_{2(\kappa)}|J|$,

$\mathbf{N}_8=(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1})-\mathbf{t}_{\kappa+1})-\beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})-\mathbf{t}_{2(\kappa)})+\Omega^*_\kappa(a_\kappa-\beta_\kappa)|J|$.

Remark 2: Assume that there are piecewise-defined bounding functions $\mathcal{G}^l_\kappa(\mathbf{s})$ and $\mathcal{G}^u_\kappa(\mathbf{s})$ that satisfy the condition $\mathcal{G}^l_\kappa(\mathbf{s})\leq\mathbf{t}_\kappa\leq\mathcal{G}^u_\kappa(\mathbf{s})$ for all $\kappa\in M_{M-1}$ and all x in the domain. Provided the recurrent IFS parameters are chosen approximately, the cubic recurrent rational fractal interpolation function with positive scaling factors will stay within these bounds.

$$(i)\ 0 \le \beta_\kappa < \min\left[a_\kappa, \frac{\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)}{\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)})}, \frac{\mathbf{t}_{\kappa+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})}{\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})}, \frac{\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa}{\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}}, \frac{\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}}{\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}}\right],$$

$$(ii)\ r_\kappa > \max\left[0, \frac{\mathbf{N}_1}{\mathbf{N}_2}, \frac{\mathbf{N}_3}{\mathbf{N}_4}, \frac{\mathbf{N}_5}{\mathbf{N}_6}, \frac{\mathbf{N}_7}{\mathbf{N}_8}\right],$$

where,

$$\mathbf{N}_1 = -[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)})) + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + \Omega_\kappa(\beta_\kappa - a_\kappa)|J|],$$
$$\mathbf{N}_2 = (\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}))],$$
$$\mathbf{N}_3 = -[(\mathbf{t}_{\kappa+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|],$$
$$\mathbf{N}_4 = (\mathbf{t}_{\kappa+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1}))) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)})) + \Omega_\kappa(\beta_\kappa - a_\kappa)|J|,$$
$$\mathbf{N}_5 = -[(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}) - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + \Omega^*_\kappa(a_\kappa - \beta_\kappa)|J|],$$
$$\mathbf{N}_6 = (\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}),$$
$$\mathbf{N}_7 = -[(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)})] - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|,$$
$$\mathbf{N}_8 = (\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) + \Omega^*_\kappa(a_\kappa - \beta_\kappa)|J|,$$
and $|J| = |\mathbf{s}_{2(\kappa)} - \mathbf{s}_{1(\kappa)}|$.

Furthermore, negative scaling factors are incorporated to derive a constrained form of the fractal interpolant. Hence, our objective is to explore the limiting properties of the recurrent cubic fractal model.
Theorem 3. If $\mathcal{G}^l_\kappa(\mathbf{s}) \le \mathbf{t}_\kappa \le \mathcal{G}^u_\kappa(\mathbf{s})$ holds for all $\kappa \in M_{M-1}$ and $j \in M_M$, then the set $\{(\mathbf{s}_\kappa, \mathbf{t}_\kappa, d_\kappa) : \kappa \in M_M\}$ constitutes a Hermite dataset. The aim is to guarantee that the negative scalings of the recurrent rational fractal models, denoted by $\mathcal{G}^l_\kappa(\mathbf{s}_\kappa) = \Omega_\kappa \mathbf{s} + \mu_\kappa$ and $\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) = \Omega^*_\kappa \mathbf{s} + \mu^*_\kappa$, stay within their designated limits. It is essential to meet the following recurring IFS criteria.

$$(i)\ 0 \le \beta_\kappa < \min\left[-a_\kappa, \frac{\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)}{\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})}, \frac{\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})}{\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})}, \frac{\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa}{\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}}, \frac{\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{i+1}}{\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}}\right],$$

$$(ii)\ r_\kappa > \max\left[0, \frac{\mathbb{B}_1}{\mathbb{B}_2}, \frac{\mathbb{B}_3}{\mathbb{B}_4}, \frac{\mathbb{B}_5}{\mathbb{B}_6}, \frac{\mathbb{B}_7}{\mathbb{B}_8}\right],$$

where

$$\mathbb{B}_1 = (\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})) + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa \beta_\kappa - \Omega_\kappa a_\kappa)|J|,$$
$$\mathbb{B}_2 = 2[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}))] + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa \beta_\kappa - \Omega_\kappa a_\kappa)|J|,$$
$$\mathbb{B}_3 = -[(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|],$$
$$\mathbb{B}_4 = (\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) + (\Omega^*_\kappa \beta_\kappa - \Omega_\kappa a_\kappa)|J|,$$
$$\mathbb{B}_5 = (\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)})) - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa a_\kappa - \Omega_\kappa \beta_\kappa)|J|,$$
$$\mathbb{B}_6 = 2[(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)})] - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa a_\kappa - \Omega_\kappa \beta_\kappa)|J|,$$
$$\mathbb{B}_7 = -[(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)})|J|],$$
and $\mathbb{B}_8 = (\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) + (\Omega^*_\kappa a_\kappa - \Omega_\kappa \beta_\kappa)|J|)$.

Proof: Here the scaling β_κ satisfies $-a_\kappa < \beta_\kappa < 0$. The first inequality (6.1) can be written in the following form:

$$\Omega_\kappa(a_\kappa \mathbf{s} + b_\kappa) + \mu_\kappa \leq \beta_\kappa f(\mathbf{s}) + \frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)}.$$

$$\Rightarrow \beta_\kappa f(\mathbf{s}) + \frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)} - [\Omega_\kappa(a_\kappa \mathbf{s} + b_\kappa) + \mu_\kappa] \geq 0.$$

Multiplying the above inequality by the positive quantity $\mathbb{Q}_\kappa(\sigma_j)$, we obtain

$$\Rightarrow \beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) - [\Omega_\kappa(a_\kappa \mathbf{s} + b_\kappa) + \mu_\kappa]\mathbb{Q}_\kappa(\sigma_j) \geq 0.$$

Using the assumed upper bound on the function f, namely $f(\mathbf{s}) \leq \Omega^*_\kappa \mathbf{s} + \mu^*_\kappa$, and noting that $\beta_\kappa > 0$, we get

$$f(\mathbf{s}) \leq \Omega^*_\kappa \mathbf{s} + \mu^*_\kappa \Rightarrow \beta_\kappa f(\mathbf{s}) \geq \beta_\kappa(\Omega^*_\kappa \mathbf{s} + \mu^*_\kappa) \Rightarrow \beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) \geq \beta_\kappa(\Omega^*_\kappa \mathbf{s} + \mu^*_\kappa)\mathbb{Q}_\kappa(\sigma_j).$$

Substituting this estimate into the previous inequality yields

$$\Rightarrow \{\beta_\kappa(\Omega^*_\kappa \mathbf{s} + \mu^*_\kappa) - [\Omega_\kappa(a_\kappa \mathbf{s} + b_\kappa) + \mu_\kappa]\}\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) \geq 0.$$

Next, expressing $\mathbf{s}$ in terms of the local parameterization $\mathbf{s} = \mathbf{s}_{1(\kappa)} + \sigma_j|J|$, we rewrite the inequality as

$$\Rightarrow \{\beta_\kappa(\Omega^*_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + \mu^*_\kappa) - [\Omega_\kappa(a_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + b_\kappa) + \mu_\kappa]\}\mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) \geq 0.$$

Separating the constant and σ_j-dependent terms, we obtain

$$\begin{aligned}\Rightarrow &[\beta_\kappa(\Omega^*_\kappa \mathbf{s}_{1(\kappa)} + \mu^*_\kappa) - (\Omega(a_\kappa \mathbf{s}_{1(\kappa)} + b_\kappa) + \mu_\kappa)]\mathbb{Q}_\kappa(\sigma_j)\\ &+ \{[\Omega^*_\kappa \beta_\kappa - \Omega_\kappa a_\kappa]|J|\}\sigma_j \mathbb{Q}_\kappa(\sigma_j) + \mathbb{P}_\kappa(\sigma_j) \geq 0.\end{aligned}$$

Utilizing the degree of elevation, we obtain

$$\begin{aligned}\mathbb{Q}_\kappa(\sigma_j) &= (r_\kappa - 1)(1-\sigma_j)^3 + (2r_\kappa - 1)(1-\sigma_j)^2\sigma_j + (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3,\\ \sigma_j\mathbb{Q}_\kappa(\sigma_j) &= (r_\kappa - 1)(1-\sigma_j)^2\sigma_j + r_\kappa\sigma_j^2(1-\sigma_j) + \sigma_j^3.\end{aligned}$$

We can be written as

$$\begin{aligned}&[\beta_\kappa(\Omega^*_\kappa \mathbf{s}_{1(\kappa)} + \mu^*_\kappa) - (\Omega(a_\kappa \mathbf{s}_{1(\kappa)} + b_\kappa) + \mu_\kappa)][(r_\kappa - 1)(1-\sigma_j)^3 + (2r_\kappa - 1)(1-\sigma_j)^2\sigma_j\\ &\quad + (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3] + \{[\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa]|J|\}[(r_\kappa - 1)(1-\sigma_j)^2\sigma_j + r_\kappa\sigma_j^2(1-\sigma_j)\\ &\quad + \sigma_j^3] + \{[(r_1 - 1)(\mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)})](1-\sigma_j)^3 + [(2r_\kappa - 1)(\mathbf{t}_\kappa - \beta_\kappa \mathbf{t}_{1(\kappa)})\\ &\quad + (r_\kappa - 1)\{h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J|\}](1-\sigma_j)^2\sigma_j + [(r_\kappa + 1)(\mathbf{t}_{i+1} - \beta_\kappa \mathbf{t}_{2(\kappa)})\\ &\quad - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|](1-\sigma_j)\sigma_j^2 + [(\mathbf{t}_{i+1} - \beta_\kappa \mathbf{t}_{2(\kappa)})]\sigma_j^3\} \geq 0.\end{aligned}$$

Which is simplified as,

$$\begin{aligned}&(r_\kappa - 1)[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})](1-\sigma_j)^3 + \{[2[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa))\\ &- \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}))] + h_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|]r_\kappa - [(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa))\\ &- \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})) + h_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|]\}(1-\sigma_j)^2\sigma_j\\ &+ \{r_\kappa[(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|)]\\ &+ [(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|]\}\sigma_j^2(1-\sigma_j)\\ &+ [(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|)]\sigma_j^3 \geq 0.\end{aligned}$$

Hence,

$$(i)\ 0 \leq \beta_\kappa < \min\left[-a_\kappa,\ \frac{\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)}{\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})},\ \frac{\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})}{\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})}\right],$$

$$(ii)\ r_\kappa > \max\left[0,\ \frac{\mathbb{B}_1}{\mathbb{B}_2},\ \frac{\mathbb{B}_3}{\mathbb{B}_4}\right],$$

where

$$\mathbb{B}_1 = (\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)})) + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|,$$

$$\mathbb{B}_2 = 2[(\mathbf{t}_\kappa - \mathcal{G}^l_\kappa(\mathbf{s}_\kappa)) - \beta_\kappa(\mathbf{t}_{1(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{1(\kappa)}))] + h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|,$$

$$\mathbb{B}_3 = -[(\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)}|J|],$$

$$\text{and}\ \ \mathbb{B}_4 = (\mathbf{t}_{i+1} - \mathcal{G}^l_\kappa(\mathbf{s}_{\kappa+1})) - \beta_\kappa(\mathbf{t}_{2(\kappa)} - \mathcal{G}^u_\kappa(\mathbf{s}_{2(\kappa)})) + (\Omega^*_\kappa\beta_\kappa - \Omega_\kappa a_\kappa)|J|.$$

The second inequality (6.1) can be written in the following form,

$$\beta_\kappa f(\mathbf{s}) + \frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)} \leq \Omega^*_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu^*_\kappa$$

$$\Rightarrow \beta_\kappa f(\mathbf{s}) + \frac{\mathbb{P}_\kappa(\sigma_j)}{\mathbb{Q}_\kappa(\sigma_j)} - [\Omega^*_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu^*_\kappa] \leq 0,$$

$$\Rightarrow [\Omega^*_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu^*_\kappa]\mathbb{Q}_\kappa(\sigma_j) - \beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) - \mathbb{P}_\kappa(\sigma_j) \geq 0,$$

Here, $f(\mathbf{s}) \geq \Omega_\kappa\mathbf{s} + \mu_\kappa \Rightarrow \beta_\kappa f(\mathbf{s}) \leq \beta_\kappa(\Omega_\kappa\mathbf{s} + \mu_\kappa) \Rightarrow \beta_\kappa f(\mathbf{s})\mathbb{Q}_\kappa(\sigma_j) \leq \beta_\kappa(\Omega_\kappa\mathbf{s} + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j)$,

$$\Rightarrow [\Omega^*_\kappa(a_\kappa\mathbf{s} + b_\kappa) + \mu^*_\kappa]\mathbb{Q}_\kappa(\sigma_j) - (\beta_\kappa(\Omega_\kappa\mathbf{s} + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j) - \mathbb{P}_\kappa(\sigma_j)) \geq 0,$$

$$\Rightarrow [\Omega^*_\kappa(a_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + b_\kappa) + \mu^*_\kappa]\mathbb{Q}_\kappa(\sigma_j) - \beta_\kappa(\Omega_\kappa(\mathbf{s}_{1(\kappa)} + (\sigma_j)|J|) + \mu_\kappa)\mathbb{Q}_\kappa(\sigma_j) - \mathbb{P}_\kappa(\sigma_j) \geq 0.$$

Applying the degree of elevation, we get

$$\begin{aligned}&[(\Omega^*_\kappa(a_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + b_\kappa) + \mu^*_\kappa) - \beta_\kappa(\Omega_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + \mu_\kappa)][(r_\kappa - 1)(1-\sigma_j)^3\\&+ (2r_\kappa - 1)(1-\sigma_j)^2\sigma_j + (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3] + [(\Omega^*_\kappa a_\kappa - \Omega_\kappa\beta_\kappa)|J|]\\&[(r_\kappa - 1)(1-\sigma_j)^2\sigma_j + r_\kappa\sigma_j^2(1-\sigma_j) + \sigma_j^3] - \mathbb{P}_\kappa(\sigma_j) \geq 0.\end{aligned}$$

Extending this formulation, we derive the following inequality:

$$\begin{aligned}&[(\Omega^*_\kappa(a_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + b_\kappa) + \mu^*_\kappa) - \beta_\kappa(\Omega_\kappa(\mathbf{s}_{1(\kappa)} + \sigma_j|J|) + \mu_\kappa)][(r_\kappa - 1)(1-\sigma_j)^3\\&+ (2r_\kappa - 1)(1-\sigma_j)^2\sigma_j + (r_\kappa + 1)\sigma_j^2(1-\sigma_j) + \sigma_j^3] + [(\Omega^*_\kappa a_\kappa - \Omega_\kappa\beta_\kappa)|J|][(r_\kappa - 1)(1-\sigma_j)^2\sigma_j\\&+ r_\kappa\sigma_j^2(1-\sigma_j) + \sigma_j^3] - \{[(r_1 - 1)(\mathbf{t}_\kappa - \beta_\kappa\mathbf{t}_{1(\kappa)})](1-\sigma_j)^3 + [(2r_\kappa - 1)(\mathbf{t}_\kappa - \beta_\kappa\mathbf{t}_{1(\kappa)}\\&+ (r_\kappa - 1)\{h_\kappa d_\kappa - \beta_\kappa d_{1(\kappa)}|J|\}](1-\sigma_j)^2\sigma_j + [(r_\kappa + 1)(\mathbf{t}_{i+1} - \beta_\kappa\mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1}\\&+ \beta_\kappa d_{2(\kappa)}|J|](1-\sigma_j)\sigma_j^2 + [(\mathbf{t}_{i+1} - \beta_\kappa\mathbf{t}_{2(\kappa)})]\sigma_j^3\} \geq 0.\end{aligned}$$

The validity of the above inequality depends on the satisfaction of the following conditions,

$$(iii)\ 0 \leq \beta_\kappa < \min\left[-a_\kappa,\ \frac{\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa}{\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)}},\ \frac{\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{i+1}}{\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}}\right],$$

$$(iv)\ r_\kappa > \max\left[0,\ \frac{\mathbb{B}_5}{\mathbb{B}_6},\ \frac{\mathbb{B}_7}{\mathbb{B}_8}\right],$$

where

$$\mathbb{B}_5 = (\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)})) - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa a_\kappa - \Omega_\kappa\beta_\kappa)|J|,$$

$$\mathbb{B}_6 = 2[(\mathcal{G}^u_\kappa(\mathbf{s}_\kappa) - \mathbf{t}_\kappa) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{1(\kappa)}) - \mathbf{t}_{1(\kappa)})] - h_\kappa d_\kappa + \beta_\kappa d_{1(\kappa)}|J| + (\Omega^*_\kappa a_\kappa - \Omega_\kappa\beta_\kappa)|J|,$$

$$\mathbb{B}_7 = -[(\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) - h_\kappa d_{\kappa+1} + \beta_\kappa d_{2(\kappa)})|J|],$$

$$\mathbb{B}_8 = (\mathcal{G}^u_\kappa(\mathbf{s}_{\kappa+1}) - \mathbf{t}_{\kappa+1}) - \beta_\kappa(\mathcal{G}^l_\kappa(\mathbf{s}_{2(\kappa)}) - \mathbf{t}_{2(\kappa)}) + (\Omega^*_\kappa a_\kappa - \Omega_\kappa\beta_\kappa)|J|).$$

6.1. An example of the constrained RRFIF lies between two piecewise lines:

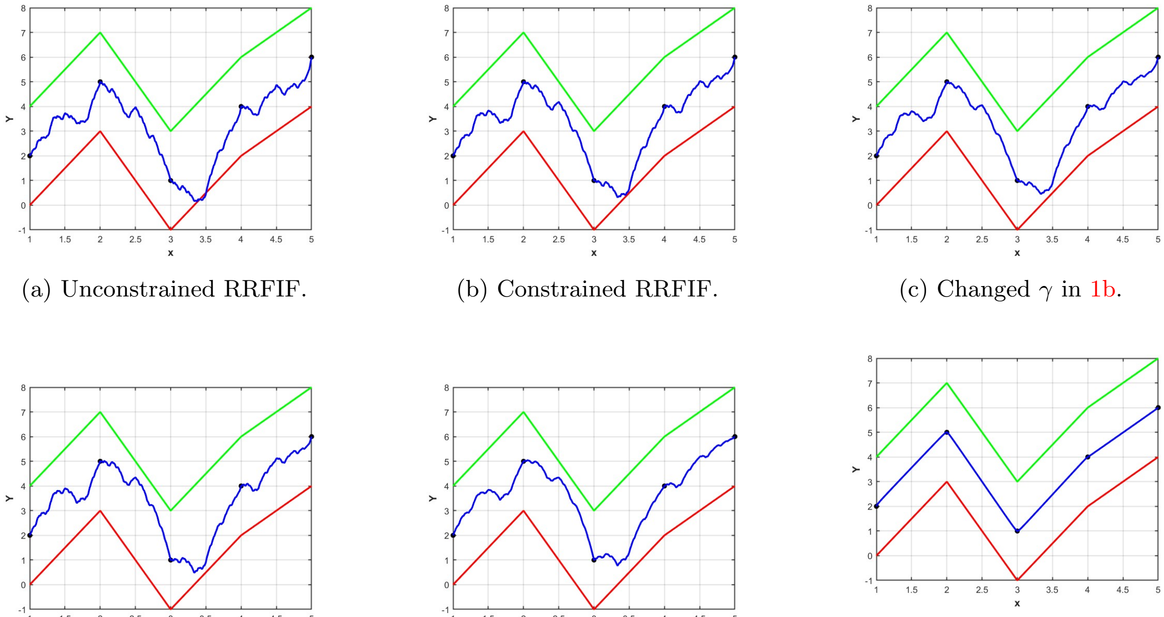
Consider the following Hermite interpolation data set:
$\{(\mathbf{s}_\kappa, \mathbf{t}_\kappa, d_\kappa) = (1,\ 2,\ 6.5), (2,\ 5,\ -0.5), (3,\ 1,\ -0.5), (4,\ 4,\ 1.5), (5,\ 6,\ 2.5)\}$. The data is between two piecewise linear functions, G^l_κ and G^u_κ, which are above and below, respectively.

$$G^l = \begin{cases} 3\mathbf{s} - 3 & \text{if} \quad 1 \leq \mathbf{s} \leq 2, \\ -4\mathbf{s} + 11 & \text{if} \quad 2 \leq \mathbf{s} \leq 3, \\ 3\mathbf{s} - 10 & \text{if} \quad 3 \leq \mathbf{s} \leq 4, \\ 2\mathbf{s} - 6 & \text{if} \quad 4 \leq \mathbf{s} \leq 5, \end{cases}$$

and

$$G^u = \begin{cases} 3\mathbf{s} + 1 & \text{if} \quad 1 \leq \mathbf{s} \leq 2, \\ -4\mathbf{s} + 15 & \text{if} \quad 2 \leq \mathbf{s} \leq 3, \\ 3\mathbf{s} - 6 & \text{if} \quad 3 \leq \mathbf{s} \leq 4, \\ 2\mathbf{s} - 2 & \text{if} \quad 4 \leq \mathbf{s} \leq 5. \end{cases}$$

Constrained RRFIFs are generated by evaluating derivative information at the prescribed grid points. Figure 1 illustrates how different parameter configurations affect the shape and behavior of the RRFIF. According to Table 1, modifying the IFS parameters alone does not guarantee that the RRFIF remains enclosed between the two piecewise linear curves, which is evident in Figure 1(a). However, when both the scaling factors and the shape parameters satisfy the conditions stated in Theorem 2, the RRFIF consistently lies within these bounds, as seen in Figure 1(b). Further, altering only the scaling parameter based on Figure 1(b) yields the RRFIF displayed in Figure 1(c), while tuning the r-shape parameter from the same reference configuration results in the curve shown in Figure 1(d). The constrained RRFIF in Figure 1(f) is obtained by adjusting the parameters with respect to the IFS values from Figure 1(b). When the scaling parameters are chosen to be zero, the restricted RRFIF reverts to the configuration depicted in Figure 1(b). Overall, Figure 1(a) illustrates the unconstrained scenario, whereas Figure 1(b) serves as the benchmark for the constrained case, with the remaining figures demonstrating the consequences of modifying individual parameter groups.



(a) Unconstrained RRFIF. (b) Constrained RRFIF. (c) Changed γ in 1b.

(d) Perturbation r in 1b. (e) Perturbation IFS in 1b. (f) Classical constrained of RRFIF in 1b.

Figure 1: RRFIF lies between piecewise lines.

Table 1: Scale control parameters and recurrence/tension parameters used in constructing the piecewise lines RRFIF in Figure 1.

Scale factor (β)	Fig.	Shape parameter (r)	Fig.
[0.32 0.49 0.31 0.48]	1a	[2 2 2 2 2]	1a
[0.31 0.45 0.32 0.45]	1b,d	[5 5 5 5 5]	1b,c
[0.29 0.40 0.29 0.40]	1c	[10 10 10 10 10]	1d
[0.25 0.39 0.28 0.38]	1e	[100 100 100 100 100]	1e
[0 0 0 0]	1f	[50 50 50 50 50]	1f

7. Performance Evaluation of Fractal Interpolation and Decision Tree Prediction

The present study utilises stock price observations from 10 consecutive trading days to evaluate the effectiveness of fractal interpolation in modelling financial time-series behaviour. Let

$$\mathbf{s} = [1, 2, \ldots, 10]$$

denote the time index (days), and let the associated closing stock values be

$$\mathbf{t} = [100,\ 102,\ 101,\ 105,\ 107,\ 106,\ 108,\ 110,\ 112,\ 115]^T.$$

This experiment aims to construct a fractal interpolation curve for the given dataset and verify whether the fractal model successfully retains the inherent price dynamics and variation patterns. In addition, the generated fractal data is employed to train a Decision Tree Regression model, and its predictive capability is evaluated using performance metrics such as Mean Squared Error (MSE), Pearson correlation, auto-correlation, and return correlation between the original and fractal-interpolated prices.

1. Mean Squared Error (MSE):

The Mean Squared Error evaluates the deviation between Decision Tree predictions and the fractal interpolated stock prices. It measures the average squared difference between the actual and predicted values, therefore indicating prediction accuracy.

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{t}_\kappa - \hat{\mathbf{t}}_\kappa)^2.$$

where $\mathbf{t}_\kappa$ = true stock price (fractal value), $\hat{\mathbf{t}}_\kappa$ = predicted stock price by the Decision Tree model, n = total number of observations. In this study, the computed MSE is **0.8500**, which confirms that the Decision Tree model achieves highly accurate predictions with minimal error. A lower MSE value implies better performance of the prediction system.

2. Pearson Correlation (Original vs Fractal Data):

Pearson correlation is used to examine whether the fractal interpolation preserves the statistical trend of the original stock prices.

$$r = \frac{\sum(\mathbf{s}-\bar{\mathbf{s}})(\mathbf{t}-\bar{\mathbf{t}})}{\sqrt{\sum(\mathbf{s}-\bar{\mathbf{s}})^2\sum(\mathbf{t}-\bar{\mathbf{t}})^2}}.$$

A value of $r = 1$ indicates a perfect linear relationship. The obtained correlation coefficient is **1.0000**, signifying that the fractal interpolation **perfectly retains the original stock price behavior without any loss of information**. This demonstrates the reliability of fractal extrapolation for time series enhancement.

3. Auto-Correlation of Fractal Data:

Auto-correlation quantifies the linear relationship between a time series and its lagged values, providing insights into the memory effect or momentum within price fluctuations.

$$\text{AutoCorr}(k) = \frac{\sum_{t=1}^{n-k}(\mathbf{T}_t-\bar{\mathbf{T}})(\mathbf{T}_{t+k}-\bar{\mathbf{T}})}{\sum_{t=1}^{n}(\mathbf{T}_t-\bar{\mathbf{T}})^2}.$$

where k denotes the lag value. Computed results indicate:

$$\text{Lag-1 Auto-correlation} = 0.9790, \quad \text{Lag-2 Auto-correlation} = 0.9550$$

These extremely high values confirm that the fractal interpolation retains long-term dependence and trend-following behaviour of stock prices — a crucial property for financial decision-making and forecasting.

4. Correlation Between Time and Price (Fractal Data):

This correlation verifies whether the fractal stock price consistently moves upward or downward with the progression of trading periods.

$$r_{\text{time,price}} = \text{corr}(\mathbf{S}, \mathbf{T}).$$

The calculated value is **0.9587**, which indicates a strong positive relationship between time and fractal price levels. Thus, the upward price trend is successfully preserved in the fractal representation.

5. Decision Tree Prediction Correlation:

Predictive accuracy of the Decision Tree model is evaluated using correlation between actual and predicted fractal prices.

$$r = \text{corr}(\mathbf{T}, \hat{\mathbf{T}}).$$

The correlation value obtained is **0.9751**, highlighting that the predicted price movement closely follows the true fractal price dynamics. This proves the suitability of machine learning algorithms for stock prediction when trained on fractal-enhanced data.

6. R-Squared (Coefficient of Determination):

The R-squared value explains the proportion of variation in stock prices accurately captured by the Decision Tree regression model.

$$R^2 = 1 - \frac{\sum(\mathbf{T}-\hat{\mathbf{T}})^2}{\sum(\mathbf{T}-\bar{\mathbf{T}})^2}.$$

The computed value is **0.9508**, implying that **95.08% of price variability is modeled correctly** by the Decision Tree. This reflects a highly efficient and data-adaptive prediction framework.

7. Correlation Between Price Changes (Returns):
Returns represent the change in stock price from one trading point to the next:

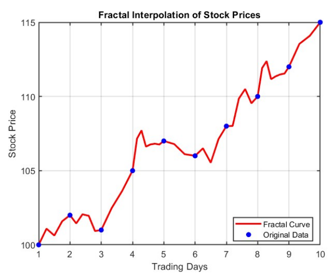
$$R_t = \mathbf{T}_t - \mathbf{T}_{t-1}.$$

The relationship between original and fractal-based returns is measured as:

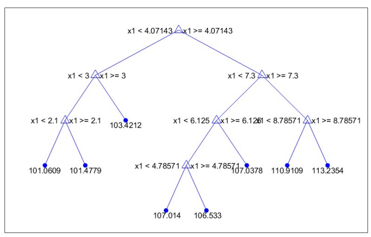
$$\text{corr}(R_{\text{original}}, R_{\text{fractal}}).$$

The computed value is **1.0000**, indicating that fractal interpolation **perfectly preserves the direction and magnitude of market movements**. This is especially important for volatility, technical trading, and short-term investment analysis.

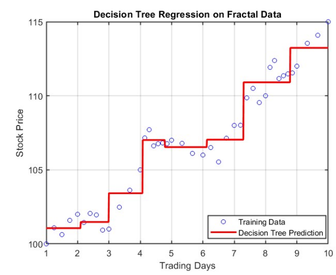
Final Interpretation and Finding:
Fractal interpolation significantly enhances the resolution of stock time-series while retaining its original characteristics. Pearson correlation of 1.0000 proves that no information distortion occurs during the interpolation process. High lag autocorrelation confirms the preservation of market momentum and memory. The decision tree model shows strong predictive capability with $R^2 = 0.9508$ and correlation of 0.9751. Return behavior is completely preserved, demonstrating suitability for financial analytics and trading-based models.



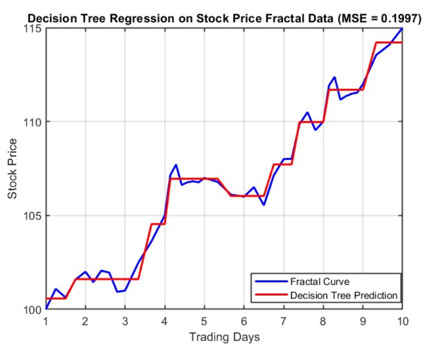
(a) Fractal curve.



(b) Decision Tree.



(c) Decision tree regression on fractal data.



(d) Fractal vs decision tree.

(e) Original vs fractal vs tree prediction.

Figure 2: RRFIF and decision tree fit.

8. Conclusion

This work introduced a Constrained Recurrent Cubic Fractal Model that combines the advantages of the Recurrent Iterated Function System with a constrained piecewise linear formulation to capture complex patterns in real data. By incorporating rational cubic and quadratic representations, the model ensures smoothness while preserving interpolation accuracy. The integration of decision tree regression further strengthens the analytical and predictive capability of the framework, allowing both precise fitting of historical data and dependable forecasting of unseen values. Experimental results on real datasets, including stock market data, demonstrate that the proposed model maintains structural consistency, adapts effectively to underlying trends, and delivers stable performance across varying conditions. Overall, the study establishes a robust and flexible fractal-based methodology for data modeling and prediction, opening new directions for future developments in computational fractal analysis and intelligent forecasting systems.

Data Availability Statement:
The datasets utilized in this study are small in size, randomly selected, and not publicly accessible. However, the complete MATLAB code and supplementary materials can be provided upon reasonable request to the corresponding author.

Authors' Contributions:
All authors made equal contributions to the conceptualization, methodology, and writing of this manuscript. Each author has reviewed and approved the final version.

Compliance with Ethical Standards

Conflict of Interest: The authors state that there are no financial, academic, or personal interests that could have influenced the outcomes of this study.

Research Involving Human Participants and/or Animals: The present work is entirely theoretical in nature and does not involve human subjects, animals, or related experimental data.

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