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A Portrayal of Integer Solutions to Non-homogeneous Quinary Nonic Equation $x^4 - y^4 = 10\left(z^2 - w^2\right)P^7$

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ABSTRACT: This paper aims at determining non-zero distinct integer solutions to Quinary nonhomogeneous polynomial equation of degree nine given by $x^4 - y^4 = 10\left(z^2 - w^2\right)P^7$. Varieties of integer solutions are obtained through utilizing substitution technique and factorization method.

Keywords: Non-homogeneous equation, quinary nonic equation, integer solutions, substitution technique, factorization method.

Contents

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of many mathematicians since antiquity. For example, in [1-8], the authors are analyzed quadratic, cubic equations with multiple variables. In [9, 10], quintic equations have been considered for obtaining many integer solutions. In [11-15], the sixth degree equations with multiple variables have been discussed for various choices of integer solutions. In [16], a Heptic equation with three unknowns has been studied for finding its integer solutions. In [17,18], an octic equation has been presented with different sets of integer solutions. The above problems motivated us to get integer solutions for another interesting ninth degree equation with five unknowns given by $x^4 - y^4 = 10\left(z^2 - w^2\right)P^7$. Substitution technique and factorization method are utilized to determine the same.

2. Method of analysis

The non-homogeneous quinary nonic equation to be solved is

$$x^4 - y^4 = 10\left(z^2 - w^2\right)P^7 \tag{2.1}$$

Introduction of the transformations

$$x = u + v, y = u - v, z = 2u + v, w = 2u - v, (u, 2u \neq \pm v) \tag{2.2}$$

in (2.1) leads to the non-homogeneous ternary heptic equation

$$u^2 + v^2 = 10P^7 \tag{2.3}$$

The process of getting varieties of integer solutions to (2.1) through solving (2.3) is presented below:

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Process 2.1

Choosing

$$u = 10^4 kR^3, P = 10R \tag{2.4}$$

in (2.3), one obtains

$$v^2 = 10^8 R^6 \left(R - k^2\right)$$

which is satisfied by

$$\begin{aligned} R &= \left(s^2 + 1\right) k^2 \\ v &= 10^4 k^7 \; s \left(s^2 + 1\right)^3 \end{aligned} \tag{2.5}$$

Using (2.5) in (2.4), we have

$$\begin{aligned} u &= 10^4 k^7 \left(s^2 + 1\right)^3 \\ P &= 10 \left(s^2 + 1\right) k^2 \end{aligned} \tag{2.6}$$

In view of (2.2), one has

$$\begin{aligned} x &= 10^4 k^7 \left(s^2 + 1\right)^3 (1 + s) \\ y &= 10^4 k^7 \left(s^2 + 1\right)^3 (1 - s) \\ z &= 10^4 k^7 \left(s^2 + 1\right)^3 (2 + s) \\ w &= 10^4 k^7 \left(s^2 + 1\right)^3 (2 - s) \end{aligned} \tag{2.7}$$

Thus, the values of x, y, z, w, P given by (2.7) & (2.6) satisfy (2.1).

Note 2. 1

Consider

$$v = 10^4 R^3 k, P = 10R$$

in (2.3). Performing the procedure as above, the corresponding integer solutions to (2.1) are given by

$$\begin{aligned} x &= 10^4 k^7 \left(s^2 + 1\right)^3 (s + 1) \\ y &= 10^4 k^7 \left(s^2 + 1\right)^3 (s - 1) \\ z &= 10^4 k^7 \left(s^2 + 1\right)^3 (2 s + 1) \\ w &= 10^4 k^7 \left(s^2 + 1\right)^3 (2 s - 1) \end{aligned}$$

together with Pin (2.6).

Process 2.2

The choice

$$v = P^3 k \tag{2.8}$$

in (2.3) gives

$$u^2 = P^6 \left(10P - k^2\right) \tag{2.9}$$

which is satisfied by

$$P=\left(10s^2-14s+5\right)k^2$$
$$u=\left(10s^2-14s+5\right)^3k^7(10s-7) \tag{2.10}$$

In view of (2.8), we have

$$v=\left(10s^2-14s+5\right)^3k^7 \tag{2.11}$$

Using (2.10) & (2.11) in (2.2), we get

$$\begin{aligned} \mathrm{x}&=\left(10\ \mathrm{s}^2-14\ \mathrm{s}+5\right)^3\mathrm{k}^7(10\ \mathrm{s}-6)\\ \mathrm{y}&=\left(10\ \mathrm{s}^2-14\ \mathrm{s}+5\right)^3\mathrm{k}^7(10\ \mathrm{s}-8)\\ \mathrm{z}&=\left(10\ \mathrm{s}^2-14\ \mathrm{s}+5\right)^3\mathrm{k}^7(20\ \mathrm{s}-13)\\ \mathrm{w}&=\left(10\ \mathrm{s}^2-14\ \mathrm{s}+5\right)^3\mathrm{k}^7(20\ \mathrm{s}-15) \end{aligned} \tag{2.12}$$

Thus, the values of $\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w},\mathrm{P}$ given by (2.12) & (2.10) satisfy (2.1).

Note 2.2

Observe that (2.9) is also satisfied by

$$P=\left(10s^2-6s+1\right)k^2$$
$$u=\left(10s^2-6s+1\right)^3k^7(10s-3) \tag{2.13}$$

and from (2.8)

$$v=\left(10s^2-6s+1\right)^3k^7 \tag{2.14}$$

Using (2.13) & (2.14) in (2.2), we get

$$\begin{aligned} x&=\left(10s^2-6s+1\right)^3k^7(10s-2)\\ y&=\left(10s^2-6s+1\right)^3k^7(10s-4)\\ z&=\left(10s^2-6s+1\right)^3k^7(20s-5)\\ w&=\left(10s^2-6s+1\right)^3k^7(20s-7) \end{aligned} \tag{2.15}$$

Thus, the values of $\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w},\mathrm{P}$ given by (2.15) & (2.13) satisfy (2.1).

Note 2.3

Consider

$$\mathrm{u}=\mathrm{P}^3\mathrm{k}$$

in (2.3). Performing the procedure as above, the corresponding two sets of integer solutions to (2.1) are given below:

Set 2.1

$$\begin{aligned} x&=\left(10s^2-14s+5\right)^3k^7(10s-6),\\ y&=\left(10s^2-14s+5\right)^3k^7(8-10s),\\ z&=\left(10s^2-14s+5\right)^3k^7(10s-5),\\ w&=\left(10s^2-14s+5\right)^3k^7(9-10s) \end{aligned}$$

together with P in (2.10).

Set 2.2

$$x=\left(10\ s^2-6\ s+1\right)^3 k^7(10\ s-2)$$
$$y=\left(10\ s^2-6\ s+1\right)^3 k^7(4-10\ s)$$
$$z=\left(10\ s^2-6\ s+1\right)^3 k^7(10\ s-1)$$
$$w=\left(10\ s^2-6\ s+1\right)^3 k^7(5-10\ s)$$

together with P in (2.13).

Process 2.3

Assume

$$P=a^2+b^2 \tag{2.16}$$

Consider the integer 10 on the R.H.S. of (2.3) as

$$10=(3+i)(3-i) \tag{2.17}$$

Substitute (2.16) & (2.17) in (2.3). Utilizing the method of factorization and equating the positive factors , we have

$$\begin{aligned} u+iv &= (3+i)(a+ib)^7 \\ &= (3+i)[f(a,b)+ig(a,b)] \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} f(a,b) &= a^7-21a^5b^2+35a^3b^4-7ab^6 \\ g(a,b) &= 7a^6b-35a^4b^3+21a^2b^5-b^7 \end{aligned} \tag{2.19}$$

On equating the real and imaginary parts in (2.18), we have

$$u=3f(a,b)-g(a,b)$$
$$v=f(a,b)+3g(a,b)$$

In view of (2.2), we get

$$\begin{aligned} x &= 4f(a,b)-2g(a,b), \\ y &= 2f(a,b)-4g(a,b), \\ z &= 7f(a,b)+g(a,b), \\ w &= 5f(a,b)-5g(a,b). \end{aligned} \tag{2.20}$$

Thus , (2.20) & (2.16) satisfy (2.1).

Note 2.4

Apart from (2.17), the integer 10 is expressed as

(i) $10=(1+3i)(1-3i)$

(ii) $10=\frac{[F(p,q)+iG(p,q)][F(p,q)-iG(p,q)]}{(p^2+q^2)^2}$

$$F(p,q)=3\left(p^2-q^2\right)+(2pq)$$
$$G(P,Q)=3(2pq)-\left(p^2-q^2\right)$$

Following the above procedure, many more integer solutions are obtained.

Process 2.4

Write (2.3) as

$$\mathrm{u}^2+\mathrm{v}^2=10\mathrm{P}^7*1 \tag{2.21}$$

Assume

$$\mathrm{P}=\left(\mathrm{r}^2+\mathrm{s}^2\right)^2\left(\mathrm{a}^2+\mathrm{b}^2\right) \tag{2.22}$$

Consider the integer 1 on the R.H.S. of (2.21) as

$$1=\frac{\left(r^2-s^2+i2rs\right)\left(r^2-s^2-i2rs\right)}{\left(r^2+s^2\right)^2} \tag{2.23}$$

Substitute (2.17), (2.22), (2.23) in (2.21). Employing the factorization method and equating the coefficients of corresponding terms , we have

$$\begin{aligned} u+iv &= (3+i)\left(r^2+s^2\right)^7\left(f(a,b)+ig(a,b)\right)\frac{\left(r^2-s^2+i2rs\right)}{\left(r^2+s^2\right)} \\ &= \left(r^2+s^2\right)^6\left(f(a,b)+ig(a,b)\right)\left\{\left[3\left(r^2-s^2\right)-2rs\right]+i\left[r^2-s^2+6rs\right]\right\} \end{aligned}$$

On equating the real and imaginary parts, we get

$u=\left(r^2+s^2\right)^6\left\{f(a,b)\left[3\left(r^2-s^2\right)-2rs\right]-g(a,b)\left[r^2-s^2+6rs\right]\right\}$

$v=\left(r^2+s^2\right)^6\left\{f(a,b)\left[r^2-s^2+6rs\right]+g(a,b)\left[3\left(r^2-s^2\right)-2rs\right]\right\}$

From (2.2), one has

$$\begin{aligned} x &= \left(r^2+s^2\right)^6\left\{f(a,b)\left[4\left(r^2-s^2\right)+4rs\right]+g(a,b)\left[2\left(r^2-s^2\right)-8rs\right]\right\}, \\ y &= \left(r^2+s^2\right)^6\left\{f(a,b)\left[2\left(r^2-s^2\right)-8rs\right]-g(a,b)\left[4\left(r^2-s^2\right)+4rs\right]\right\}, \\ z &= \left(r^2+s^2\right)^6\left\{f(a,b)\left[7\left(r^2-s^2\right)+2rs\right]+g(a,b)\left[\left(r^2-s^2\right)-14rs\right]\right\}, \\ w &= \left(r^2+s^2\right)^6\left\{f(a,b)\left[5\left(r^2-s^2\right)-10rs\right]-g(a,b)\left[5\left(r^2-s^2\right)+10rs\right]\right\}. \end{aligned} \tag{2.24}$$

Thus, (2.22) & (2.24) satisfy (2.1).

3. Conclusion

In this paper, an attempt has been made to obtain many integer solutions to nonhomogeneous Diophantine equation of degree nine with five unknowns. Substitution strategy and factorization technique are utilized for finding the required integer solutions to the ninth degree equation with five unknowns. In this analysis, the given equation is reduced to lower degree equation for which the integer solutions can be found elegantly.

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