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Sine Odd Perks-G family of Distributions: Properties, Estimation Techniques and Applications with Insurance and Failure Data

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ABSTRACT: This paper introduces a new trigonometric family of distributions which is named as "Sine Odd Perks-G family". Few submodels of the family are proposed including one based on the Lomax distribution named "Sine Odd-Perks Lomax" distribution. We derive its basic statistical properties and illustrate its behavior through density plots showing right skewed, left skewed, n-shape and symmetrical curves and hazard rate plots capturing shapes such as increasing, decreasing, inverted bathtub, sigmoid, and inverted V-shape. Three-dimensional skewness and kurtosis plots are also visualized. The model parameters are estimated using different estimation methods, and the performance of the estimators is evaluated through Monte carlo simulation study. We also examined the utility of the distribution through actuarial data and failure times data.

Keywords: Perks distribution, Odd Perks-G family, Hazard rate, Sine-G family, actuarial data.

Contents

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1. Introduction

In recent decades, statistical modeling has witnessed a growing emphasis on the development of generalized families of distributions to provide greater flexibility in modeling real-life data. It has become evident that classical distributions, such as exponential, gamma, and Weibull, often fail to capture the complex behavior found in modern datasets, especially those involving varying hazard rate shapes and heavy tails. These limitations have inspired the formulation of numerous generalized families through compounding, transformation, and parameter induction techniques. Some of the most widely recognized generalized distributions include the exponentiated Weibull family by Mudholkar and Srivastava [3], the Exponential-G family by Gupta et al. [4], especially for modeling failure rate lifetime data. A new method of adding parameters introduced by Marshall & Olkin [5], A new method for generating families of continuous distributions(T-X family) introduced by Alzaatreh et al. [6], Beta generated family by Eugene et al. [7], Weibull-G family by Bourguignon et al. [8], Gamma-G family by Zografos et al. [9]. One such development is the Odd Perks-G (OP-G) family [10], which emerged as a flexible generalization of the Perks distribution [11] through an odd-based transformation. The OP-G family has shown improved performance in fitting diverse types of lifetime data and is known for its adaptability in terms of skewness, kurtosis, and hazard rate shapes. Motivated by the importance of modeling failure and mortality rates we extend the recently introduced OP-G family by proposing the new "Sine Odd Perks-G" family, and illustrate its applicability using real-life datasets on including insurance data. More recently, trigonometric transformations, particularly sine and cosine functions, have been used to introduce additional flexibility into baseline distributions, giving rise to various sine-generalized families. Early contribution to the distributions using trigonometric functions done by G.K Gilberts [12] in 1892. Fisher used sine and cosine functions together to derive Fisher distribution [21] in 1953. Later, Kumar et al. introduced the sine-G family [14]. Some other notable sine generated families include, a new sine-G family by Mahmood Zafar et al. [15], sine-kumaraswamy G family by Chesneau and Jamal [16], sine Topp-leone G family by Al-Bbtain et al. [17], Transmuted sine-G family by Sakthivel et al. [19], sine generalized-G family by Oramulu et al. [18]. These trigonometric families allow for the introduction of additional shape behavior without significantly complicating the mathematical form of the model. Inspired by this methodological approach, we propose a new and flexible trigonometric family of distributions called the "SOP-G" family, which combines the structural features of the odd-Perks generator with the added flexibility imparted by sine transformation.

In general, the primary objective of proposing new probability distributions is to develop flexible models capable of effectively capturing complex non-normal data patterns. This flexibility is often achieved by introducing additional parameters, usually location, scale, or shape parameters. In this direction, numerous extensions of the well-known Lomax distribution have been proposed to enhance its modeling capabilities. In developing the SOP-G family, we base our model on the Lomax distribution which is a pareto type-II distribution given by Lomax [20] in 1954, and widely recognized for its ability to model heavy-tailed and positively skewed data commonly encountered in reliability and survival analysis. By applying the SOP generator to the Lomax baseline, we derive a new and flexible model called "Sine odd-Perks Lomax" (SOPL) distribution. This model retains the heavy tailed characteristics of the Lomax while extending its applicability with additional parameterization through OP-G family and the use of sine generator making the model well suited for modeling real-world failure time data that may be challenging to fit with existing distributions. The sine-G family [14] is derived with the cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ as

$$F(x) = \sin\left[\frac{\pi}{2}W(x)\right] \quad ; \quad f(x) = \frac{\pi}{2}w(x)\cos\left[\frac{\pi}{2}W(x)\right], \quad x \in \mathbb{R} \tag{1.1}$$

respectively. where $W(x)$ and $w(x)$ denotes the cdf and pdf of a parent distribution respectively.

The OP-G family is introduced by Elbatal et al. [10], cdf and pdf of this family for any baseline distribution G(x; θ) are given below in (1.2).

$$F_{opg}(x;\alpha,\beta,\theta) = 1 - \frac{1+\alpha}{1+\alpha e^{\beta\left[G(x;\theta)/\bar{G}(x;\theta)\right]}} \quad ; \quad f_{opg}(x;\alpha,\beta,\theta) = \frac{\alpha\beta\,(1+\alpha)\,g(x;\theta)\,e^{\beta\left(\frac{G(x;\theta)}{\bar{G}(x;\theta)}\right)}}{\bar{G}(x;\theta)^2\,\left[1+\alpha\,e^{\beta\,\left(\frac{G(x;\theta)}{\bar{G}(x;\theta)}\right)}\right]^2}. \tag{1.2}$$

where θ is the vector of the parameters of the baseline distribution. To demonstrate the usefulness of the SOPL model, we apply it to two real-world datasets: (i) Theft rates (crime related data) and (ii) Insurance data. The varying distributional patterns exhibited by these datasets highlight the practical usefulness of the SOP-G family.

The rest of the paper proceeds as follows. Section 2 presents the newly proposed Sine odd-Perks G family along with its fundamental functions and statistical properties. Five different estimation approaches were utilized in section 3. section 4 introduces few sub-models derived from the SOP-G family. In section 5, we develop the Sine Odd-Perks Lomax (SOPL) distribution, describe its properties and section 6 reports the results of a simulation study, evaluating the performance of the estimators across different estimation techniques. Section 7 illustrates the practical applicability of the model through real-world data analysis. Finally, section 8 presents concluding remarks and outlines directions for future research.

2. Sine Odd Perks-G Family

By imposing the cdf of OP-G family (1.2) into the trigonometric transformation of the sine-G family (1.1), we have obtained a new class of continuous univariate distributions called "Sine Odd Perks-G family of distributions", which has cdf and pdf as

$$F_{sop}(x;\alpha,\beta,\theta) = \cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha e^{\beta\left[G(x;\theta)/\bar{G}(x;\theta)\right]}}\right)\right\}; \qquad x>0. \quad \& \tag{2.1}$$

$$f_{sop}(x;\alpha,\beta,\theta) = \frac{\pi}{2}\alpha\beta\frac{(1+\alpha)}{\left(1+\alpha e^{\beta\left[G(x;\theta)/\bar{G}(x;\theta)\right]}\right)^2}\frac{e^{\beta\left[G(x;\theta)/\bar{G}(x;\theta)\right]}}{\bar{G}(x;\theta)^2} \sin\left\{\frac{\pi}{2}\left(\frac{(1+\alpha)}{1+\alpha e^{\beta\left[G(x;\theta)/\bar{G}(x;\theta)\right]}}\right)\right\}g(x;\theta). \tag{2.2}$$

respectively. where θ is the vector of the parameters of the baseline distribution $G(x;\theta)$, further simply notated as $G(x)$ for convenience. $\alpha>0$ and $\beta>0$ are the shape parameters.

2.1. Quantile Function

The quantile function that is used to produce sample observations can be obtained by taking the inverse of the cdf of SOP-G family given as follows.

$$Q(u) = G^{-1}\left\{1+\beta\left[\log\left(\frac{1}{\alpha}\left(\frac{\pi}{2}\frac{(1+\alpha)}{\arccos(u)}-1\right)\right)\right]^{-1}\right\}^{-1}, u\in(0,1).$$

2.2. Linear Representation

Using Taylor's series expansion of cos(x) given in (2.3), the cdf of the SOP-G family is given as

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \tag{2.3}$$

$$F_{sop}(x;\alpha,\beta,\theta) = \left(\frac{\pi}{2}\right)^{2n}\sum_{i=1}^{n}\frac{(-1)^n}{(2n)!}\frac{(1+\alpha)^{2n}}{\left(1+\alpha\, e^{\beta[G(x)/\bar{G}(x)]}\right)^{2n}} \quad ; \quad x>0,\alpha,\beta>0\,.$$

After applying binomial expansions, we get $F_{sop}(x;\alpha,\beta,\theta) = \eta_{p,t}\sum_{p=0}^{\infty}G(x)^{k+l}$. Differentiating the cdf, we get the pdf of SOP-G family as follows.

$$f_{sop}(x;\alpha,\beta,\theta) = (k+l)\sum_{p=0}^{\infty}\eta_{p,t}\,G(x)^{k+l-1}\,g(x)\,. \tag{2.4}$$

where $p=\{n,i,j,k,l\}$, $t=\{\alpha,\beta\}$ and $\eta_{p,t} = \left(\frac{\pi}{2}\right)^{2n}\binom{2n}{i}\binom{-2n}{j}\binom{-k}{l}(\beta j)^k\frac{(-1)^l\alpha^{i+j}}{k!}$.

2.3. Moments

By using the general formula, the r^{th} ordinary moment of SOP-G family is given as

$$\mu_r' = \int_0^\infty x^r\, f_{sop}(x;\alpha,\beta,\theta)\, dx. = (k+l)\,\eta_{p,t}\int_0^\infty x^r G(x;\theta)^{k+l-1}\, g(x;\theta)\, dx. \quad (2.5)$$

for $r = 0, 1, 2, 3\ldots$

3. Estimation Techniques

This section examines various estimation techniques to estimate the parameters α and β of the proposed SOP-G family.

3.1. Maximum Likelihood Estimation

Consider $x_1, x_2, x_3, \ldots x_n$ to be 'n' independently and identically distributed random variables of the SOP-G family having the parameters α, β has a likelihood function given in equation(3.1).

$$L(x) = \left(\frac{\pi}{2}\alpha\beta\right)^n \frac{(1+\alpha)^n}{\prod_{i=1}^n\left[1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}\right]^2}\frac{exp(\beta\sum_{i=1}^n[G(x_i)/\bar{G}(x_i)])}{\prod_{i=1}^n[1-G(x_i)]^2}$$
$$\prod_{i=1}^n \sin\left\{\frac{\pi}{2}\left(\frac{(1+\alpha)}{1+\alpha\, exp(\beta\sum_{i=1}^n[G(x_i)/\bar{G}(x_i)])}\right)\right\} g(x_i;\theta). \quad (3.1)$$

Applying logarithm we will get the log-likelihood function as follows

$$\log L(x) = n\log(\frac{\pi}{2}\alpha\beta) + n\log(1+\alpha) + \beta\sum_{i=1}^n\left(\frac{G(x_i)}{1-G(x_i)}\right) - 2\sum_{i=1}^n \log\left(1+\alpha\, e^{[G(x_i)/(1-G(x_i))]}\right)$$
$$-2\sum_{i=1}^n \log(1-G(x_i)) + \sum_{i=1}^n \log\left[\sin\left\{\frac{\pi}{2}\frac{(1+\alpha)}{1+\alpha\, e^{[G(x_i)/(1-G(x_i))]}}\right\}\right] + \sum_{i=1}^n \log G(x_i).$$

The partial derivation of log-likelihood function with respect to the parameter α is

$$\frac{\partial \log L(x)}{\partial\alpha} = \frac{n}{\alpha} + \frac{n}{1+\alpha} - 2\sum_{i=1}^n \frac{e^{\beta[G(x_i)/\bar{G}(x_i)]}}{1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}} + \sum_{i=1}^n \cot\left\{\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}}\right\}$$
$$\left[\frac{\pi}{2}\frac{1-e^{\beta[G(x_i)/\bar{G}(x_i)]}}{\left(1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}\right)^2}\right]. \quad (3.2)$$

The partial derivation of log-likelihood function with respect to the parameter β is

$$\frac{\partial \log L(x)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^n[G(x_i)/\bar{G}(x_i)] - \alpha\sum_{i=1}^n [G(x_i)/\bar{G}(x_i)]\, e^{\beta[G(x_i)/\bar{G}(x_i)]}$$
$$\left[2\left(1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}\right)^{-1} - \cot\left\{\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}}\right\}\right]. \quad (3.3)$$

Maximum likelihood estimators(MLEs) of the parameters α and β can be obtained by solving the likelihood equations given in (3.2) and (3.3).

3.2. Least Squares Estimation

The least squares function for the SOP-G family is given by

$$LSE(\alpha,\beta,\theta)=\sum_{i=1}^{n}\left[F_{sop}(X_{(i)})-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[\cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha e^{\beta[G(x_i)/\bar{G}(x_i)]}}\right)\right\}-\frac{i}{n+1}\right]^2 \tag{3.4}$$

The partial derivatives of this function with respect to the parameters α and β are given as

$$\frac{\partial LSE}{\partial\alpha}=\pi\sum_{i=1}^{n}\left(F_{sop}(X_{(i)})-\frac{i}{n+1}\right)\left[\left(\frac{e^{\beta Z_i}-1}{(1+\alpha e^{\beta Z_i})^2}\right)\sin\left(\frac{\pi}{2}\left(\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)\right)\right] \tag{3.5}$$

$$\frac{\partial LSE(\alpha,\beta,\theta)}{\partial\beta}=\pi\sum_{i=1}^{n}\left(F_{sop}(X_{(i)})-\frac{i}{n+1}\right)\left[\left(\frac{\alpha(1+\alpha)Z_i e^{\beta Z_i})}{(1+\alpha e^{\beta Z_i})^2}\right)\sin\left(\frac{\pi}{2}\left(\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)\right)\right] \tag{3.6}$$

where, $Z_i=\frac{G(X_{(i)};\theta)}{\bar{G}(X_{(i)};\theta)}$

The least squares estimators of the parameters are obtained by numerically minimizing the sum of squared errors given in equation (3.4), using the partial derivatives given in equations (3.5) and (3.6).

3.3. Maximum Product Spacing Estimation

For an ordered sample $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ from the SOP-G family, the spacings D_i's are defined using the cdf $F_{SOP}(x)$. consider the spacings as,

$$D_i(\alpha,\beta,\theta)=F_{sop}(X_{(i)})-F_{sop}(X_{(i-1)}),\quad for\, i=1,2,\ldots,n+1 \tag{3.7}$$

with the boundary conditions, $F_{sop}(X_{(0)})=0, F_{sop}(X_{(n+1)})=1$. consider the log-product of spacings,

$$MPS(\alpha,\beta,\theta)=\sum_{i=1}^{n+1}\log D_i(\alpha,\beta,\theta) \tag{3.8}$$

importing (2.1) in (3.7) we get,

$$D_i(\alpha,\beta,\theta)=\cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha e^{\beta Z_i}}\right)\right\}-\cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha e^{\beta Z_{i-1}}}\right)\right\} \tag{3.9}$$

To maximize $MPS(\alpha,\beta,\theta)$, compute derivatives with respect to the parameters α and β. Consider $\omega_i=\sin\left(\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta Z_i}}\right)$

$$\frac{\partial MPS(\alpha,\beta,\theta)}{\partial\alpha}=\sum_{i=1}^{n+1}\frac{1}{D_i}\frac{\partial D_i}{\partial\alpha}=\sum_{i=1}^{n+1}\frac{1}{D_i}\left\{\frac{\pi}{2}\left[\frac{(e^{\beta Z_i}-1)\omega_i}{(1+\alpha e^{\beta Z_i})^2}-\frac{(e^{\beta Z_i}-1)}{(1+\alpha e^{\beta Z_{i-1}})^2}\omega_{i-1}\right]\right\}\quad,$$

$$\frac{\partial MSE(\alpha,\beta,\theta)}{\partial\beta}=\sum_{i=1}^{n+1}\frac{1}{D_i}\frac{\partial D_i}{\partial\beta}=\sum_{i=1}^{n+1}\frac{1}{D_i}\left\{\frac{\pi}{2}\left[\frac{\alpha(1+\alpha)Z_i e^{\beta Z_i}\omega_i}{(1+\alpha e^{\beta Z_i})^2}-\frac{\alpha(1+\alpha)Z_{i-1}e^{\beta Z_{i-1}}}{(1+\alpha e^{\beta Z_{i-1}})^2}\omega_{i-1}\right]\right\}$$

Equating the partial derivatives to zero and find the estimators of the parameter where the function $MPS(\alpha,\beta,\theta)$ is maximum.

3.4. Cramer-von Mises Estimation

The CvM method is used to estimate the parameters α,β of the SOP-G family by minimizing the Cramer-von Mises minimum distance estimation function given as

$$CvM(\alpha,\beta,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]^2=\frac{1}{12n}+\sum_{i=1}^{n}\left[\cos\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)-\frac{2i-1}{2n}\right]^2$$

The partial derivatives of $CvM(\alpha,\beta,\theta)$ with respect to the parameters α,β are given as

$$\frac{\partial CvM(\alpha,\beta,\theta)}{\partial\alpha}=2\sum_{i=1}^{n}\left[\cos\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)-\frac{2i-1}{2n}\right]\frac{\partial F(X_{(i)})}{\partial\alpha},$$

$$\frac{\partial CvM(\alpha,\beta,\theta)}{\partial\beta}=2\sum_{i=1}^{n}\left[\cos\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)-\frac{2i-1}{2n}\right]\frac{\partial F(X_{(i)})}{\partial\beta}$$

3.5. Anderson-Darling Estimation

An approach of parameter estimation based on the Anderson-Darling statistic where parameter estimates of the SOP-G family are obtained by minimizing the Anderson-Darling function (3.10) with respect to each parameter.

$$AD(\alpha,\beta,\theta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log(F(X_{(i)}))+\log(1-F(X_{(n+1-i)}))\right]$$

$$AD(\alpha,\beta,\theta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[\cos\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)\right]+\log\left[1-\cos\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_{n+1-i}})}\right)\right]\right\} \tag{3.10}$$

Partial derivative of $AD(\alpha,\beta,\theta)$ with respect to α is

$$\frac{\partial AD(\alpha,\beta,\theta)}{\partial\alpha}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\left(\frac{1}{F(X_{(i)})}-\frac{1}{1-F(X_{(n+1-i)})}\right)\frac{\partial F(X_{(i)})}{\partial\alpha}\right] \tag{3.11}$$

Partial derivative of $AD(\alpha,\beta,\theta)$ with respect to β is

$$\frac{\partial AD(\alpha,\beta,\theta)}{\partial\beta}=-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\left(\frac{1}{F(X_{(i)})}-\frac{1}{1-F(X_{(n+1-i)})}\right)\frac{\partial F(X_{(i)})}{\partial\beta}\right] \tag{3.12}$$

where

$$\frac{\partial F(X_{(i)})}{\partial\alpha}=\frac{\pi}{2}\frac{(e^{\beta Z_i}-1)}{(1+\alpha e^{\beta Z_i})^2}\sin\left(\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta Z_i}}\right),$$

$$\frac{\partial F(X_{(i)})}{\partial\beta}=\frac{\pi}{2}\frac{\alpha(1+\alpha)Z_i e^{\beta Z_i}}{(1+\alpha e^{\beta Z_i})^2}\sin\left(\frac{\pi}{2}\frac{1+\alpha}{(1+\alpha e^{\beta Z_i})}\right)\quad;\quad Z_i=\frac{G(X_{(i)};\theta)}{\bar{G}(X_{(i)};\theta)}.$$

4. Special Models

4.1. Sine Odd Perks Exponential Distribution

Taking the exponential distribution with parameter θ as the baseline distribution, we can obtain the following cdf and pdf of the "Sine Odd Perks Exponential distribution".

$$F_{sope}(x;\alpha,\beta,\theta)=\cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha e^{\beta(e^{\theta x}-1)}}\right)\right\},\quad x>0,\alpha,\beta,\theta>0.$$

and

$$f_{sope}(x;\alpha,\beta,\theta)=\frac{\pi}{2}\alpha\beta\theta\frac{(1+\alpha)}{\left[1+\alpha e^{\lambda(e^{\theta x}-1)}\right]^2}e^{\left[\theta x+\lambda\left(e^{\theta x}-1\right)\right]}\sin\left\{\frac{\pi}{2}\frac{(1+\alpha)}{\left[1+\alpha e^{\lambda(e^{\theta x}-1)}\right]}\right\}.$$

4.2. Sine Odd Perks Weibull Distribution

We can obtain the "Sine Odd Perks Weibull Distribution" by taking the Weibull distribution as the parent distribution for the "Sine Odd Perks-G family". The cdf and pdf of the sine odd-perks weibull distribution are given as,

$$F_{sopw}(x;\alpha,\beta,\lambda,\delta)=\cos\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha\, exp\left(\beta[e^{(x/\lambda)}-1]\right)}\right)\right\};x>0,\alpha,\beta,\lambda,\delta>0.$$

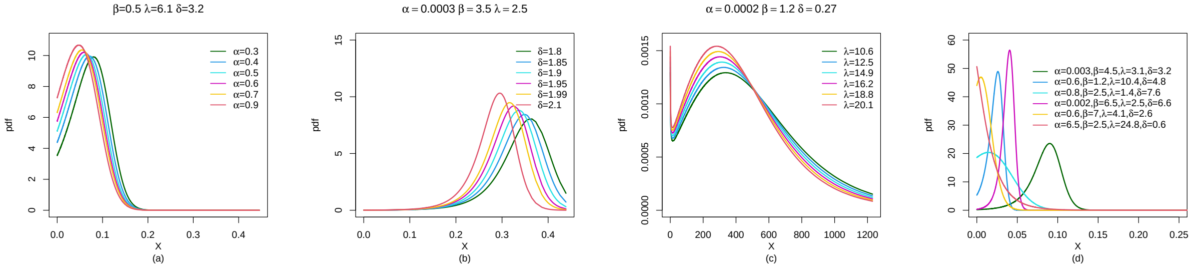


Figure 1: Density curves for SOPL distribution.

and

$$f_{sopw}(x;\alpha,\beta,\lambda,\delta) = \frac{\pi}{2}\frac{\alpha\beta\delta}{\lambda^{\delta}}x^{\delta-1}e^{\left(\frac{x}{\lambda}\right)^{\delta}}exp\left(\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\delta}-1}\right)\right)\left[\frac{1+\alpha}{\left(1+\alpha\, exp\left[\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\delta}-1}\right)\right]\right)^{2}}\right]$$
$$\sin\left\{\frac{\pi}{2}\left(\frac{1+\alpha}{1+\alpha\, exp\left[\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\delta}-1}\right)\right]}\right)\right\}.$$

respectively.

5. Sine Odd Perks Lomax Distribution

The cdf and pdf of the lomax distribution are considered as

$$G(x;\theta) = 1-(1+\lambda x)^{-\delta}, \quad x\in(0,\infty). \tag{5.1}$$

and

$$g(x;\theta) = \delta\lambda(1+\lambda x)^{-(\delta+1)}. \tag{5.2}$$

respectively. Here, $\theta = (\lambda,\delta)$ is the vector of the scale and shape parameters, respectively. The cdf and pdf of the newly proposed "Sine odd Perks Lomax distribution" can be obtained by inserting(5.1) and (5.2) into (2.1) and (2.2) respectively, and they are given as,

$$F_{sopl}(x;\alpha,\beta,\lambda,\delta) = \cos\left\{\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]}}\right\}, \quad x>0,\alpha,\beta,\lambda,\delta>0. \tag{5.3}$$

and

$$f_{sopl}(x;\alpha,\beta,\lambda,\delta) = \frac{\pi}{2}\alpha\beta\lambda\delta(1+\lambda x)^{\delta-1}e^{\beta[(1+\lambda x)^{\delta}-1]}\frac{(1+\alpha)}{(1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]})^{2}}$$
$$\sin\left\{\frac{\pi}{2}\frac{(1+\alpha)}{(1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]})}\right\}. \tag{5.4}$$

respectively. The density curves of the SOPL distribution at different values of the parameters having different shapes are presented below. From figure 1, we can observe, the shape of the SOPL density is highly sensitive to the parameter α. when $\alpha < 1$ (for very smaller values of α), SOPL produces left skewed and "n" shaped density curves. when $\alpha \approx 1$, exhibits unimodal curves and $\alpha > 1$ produces decreasing pattern. β mainly stretches the right tail, where as $\lambda > 1$ controls the peakedness of the curve. $\delta > 1$ dominating the left tail and $\delta < 1$ controls the right tail also.

5.1. Reliability Functions

Survival function and the Hazard rate function of the SOPL distribution are

$$S(x) = 1 - F_{sopl}(x;\alpha,\beta,\lambda,\delta) = 1-\cos\left\{\frac{\pi}{2}\frac{1+\alpha}{1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]}}\right\}.$$

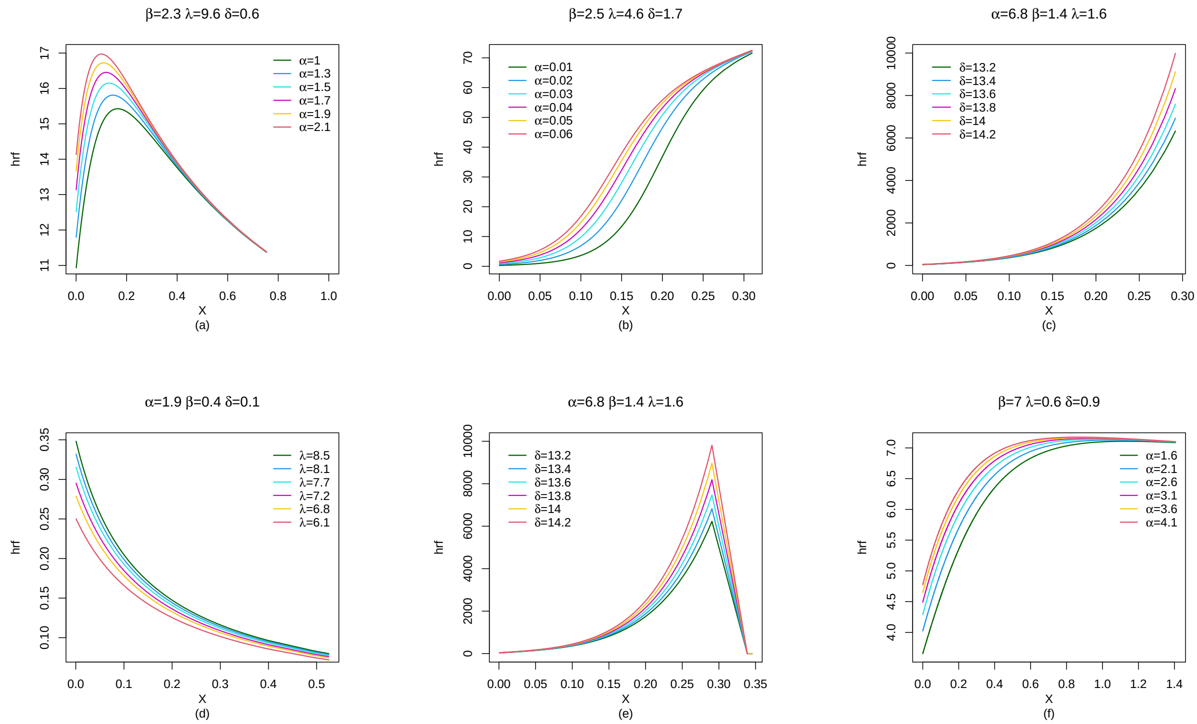


Figure 2: Hazard curves of SOPL distribution

and

$$H(x) = \frac{f_{sopl}(x;\alpha,\beta,\lambda,\delta)}{F_{sopl}(x;\alpha,\beta,\lambda,\delta)} = \frac{\pi}{2}\alpha\beta\lambda\delta(1+\lambda x)^{\delta-1}e^{\beta[(1+\lambda x)^{\delta}-1]}\frac{(1+\alpha)}{(1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]^{2}})}$$
$$\cot\left\{\frac{\pi}{4}\frac{(1+\alpha)}{(1+\alpha e^{\beta[(1+\lambda x)^{\delta}-1]})}\right\}.$$

respectively. where, $x > 0, \alpha, \beta, \lambda, \delta > 0$.

Different shapes of the hazard rate curves for the SOPL distribution at different values of the parameters are presented in the figure below.

From figure 2, Depending on the parameter values, the proposed distribution's hazard rate takes on a variety of shapes. It displays a sigmoid shape when $\alpha < 1$, an inverted bathtub form when $\delta < 1$, a decreasing hazard rate when both $\beta < 1$ and $\delta < 1$. For $\lambda < 1$, monotonically increasing hazard curve is obtained. Inverted "V" shape also obtained for all the parameters greater than one.

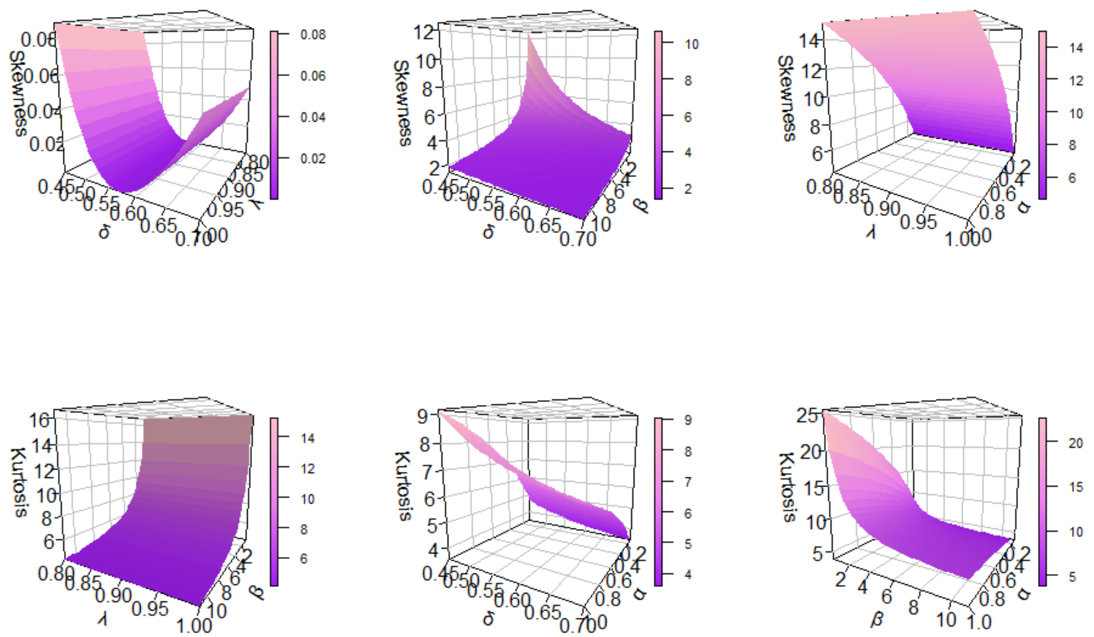
5.2. Linear Representation

Applying Taylor's series expansion of the cosine function and the binomial expansions to the cdf (5.3) of the SOPL distribution, we obtain the following expansion.

$$F(x;\alpha,\beta,\lambda,\delta) = \phi_{s,r}\, x^{m}.$$

Where as the pdf of SOPL distribution is, $f(x) = m\,\phi_{s,r}\,x^{m-1}$. where $s = \{i,j,k,l,m,n\}$, $r = \{\alpha,\beta,\lambda,\delta\}$ and

$$\phi_{s,r} = \left(\frac{\pi}{2}\right)^{2n}\sum_{s=0}^{\infty}\frac{(-1)^{n+i}(\alpha^{i+j})(\beta j)^{k}\lambda^{m}}{(2n)!k!}\binom{2n}{i}\binom{-2n}{j}\binom{k}{l}\binom{\delta(k-l)}{m}.$$



5.3. Mathematical Properties

5.3.1. Quantile Function.
The quantile function of SOPL distribution, which is used to generate values from the random variable, is stated below.

$$Q(u;\alpha,\beta,\lambda,\delta) = F_{sopl}^{-1}(x) = \frac{1}{\lambda}\left\{\left[\frac{1}{\beta}\left\{\log\left[\frac{1}{\alpha}\left(\frac{\pi}{2}\frac{(1+\alpha)}{\arccos(u)}-1\right)\right]\right\}+1\right]^{\frac{1}{\delta}}-1\right\}; \quad u \in (0,1).$$

5.3.2. Moments.
The r^{th} ordinary moment about the origin of the SOPL distribution can be obtained using,

$$\mu_r' = \int_0^\infty x^r f_{sopl}(x;\alpha,\beta,\lambda,\delta)\,dx = m\,\phi_{s,r}\int_0^\infty x^{r+m-1}\,dx$$

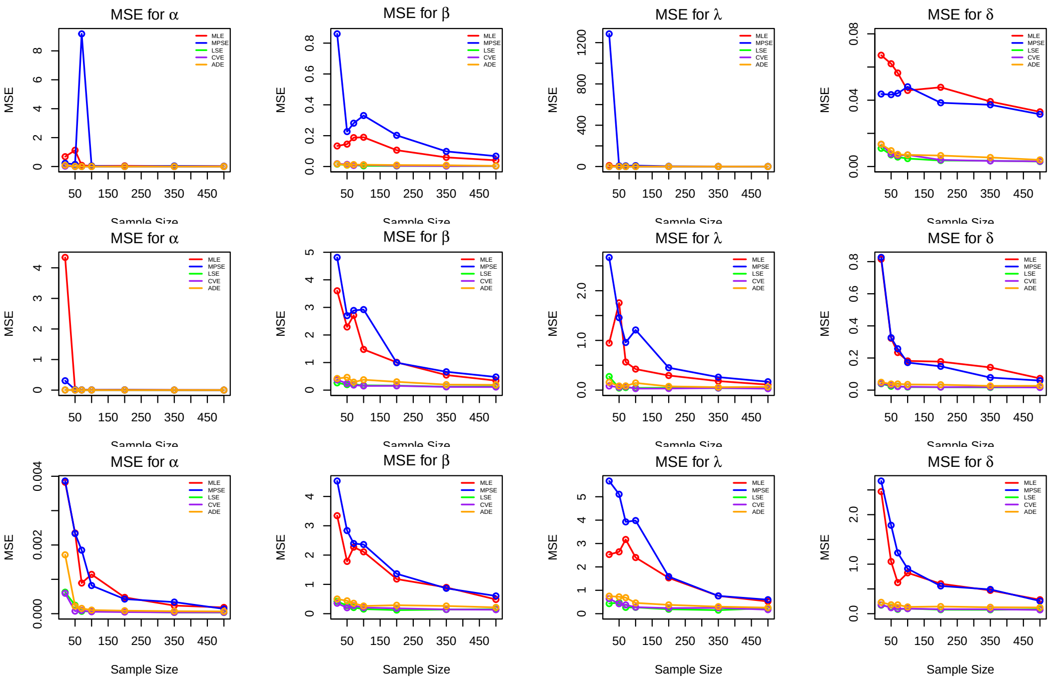
From table 1, it can be seen that higher values of the parameter α lead to a lower the first four moments and variance. At the same time, higher values of both α and β values tend to increase skewness and kurtosis, producing more asymmetric shapes and heavier tails. λ also make the distribution more peaked and increase kurtosis.

In the figure 3, we can observe, how the parameters α and β affect skewness and kurtosis of SOPL distribution. As α, β increases, both skewness and kurtosis decreases, which demonstrates substantial flexibility of SOPL distribution, allowing for highly skewed and leptokurtic shapes when α and β are small.

6. Simulation Study

To ensure reliable parameter estimation for the SOPL model, we employed multiple estimation techniques including maximum likelihood estimation (MLE), least squares estimation (LSE), maximum product spacing (MPS), Cramer-von Mises (CvM) and Anderson-Darling (AD) methods. We developed the estimation procedures for the SOPL model by specializing the general estimation framework established

Figure 4: MSE plots for the results in tables 6, 7, and 8 in each row respectively.



for the SOP-G family in section 3. We conducted a Monte-carlo simulation study with 1000 replications across multiple parameter sets and sample sizes {20,50,70,100,200,350,500}. The performance of each estimator was assessed based on mean estimates, bias and mean square error(MSE). The complete numerical findings from this analysis are presented in tables 6 through 8 in appendix. The simulation results reveals that, across different sample sizes and estimation techniques, the LSE method yields lower MSE values for most parameters of the SOPL model, indicating its superior overall performance compared to the other methods considered.

7. Applications

The proposed SOPL distribution, with its flexible hazard rate structure capable of exhibiting increasing, decreasing, and bathtub-shaped forms, demonstrates strong potential in modeling heterogeneous actuarial data related to insurance and failure times data. Well-known distributions that are compared with our proposed SOPL distribution include Sine modified-Kies Lomax (SMKL) [29], Sine Power Lomax (SPL) [30], Type-II Topp-Leone power Lomax (TIITLPL) [31], Odd Lomax - Lomax (OLL) [25], Sine Exponential (SE) [28], Sine Lomax (SL) [16], Power Lomax (PL) [26], Lomax (L) [20].

We take into account information criterion such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC). Anderson Darling (A*), Cramer-von mises (W*), and Kolmogorov-Smirnov test statistic (Dn) were also computed to evaluate the efficiency of the proposed model.

Dataset 1: The first dataset quantifies net insurance activity per 100 housing units, showing how many policies are maintained or newly issued. It reflects demand and availability of standard market insurance, https://instruction.bus.wisc.edu/jfrees/jfreesbooks/regression%20modeling/bookwebdec2010/data.html. The data are 5.3, 3.1, 4.8, 5.7, 5.9, 4, 7.9, 6.9, 7.6, 3.1, 1.3, 14.3, 12.1, 10.9, 10.7, 13.8, 8.9, 11.5, 8.5, 5.2, 2.7, 1.2, 0.8, 5.2, 1.8, 2.1, 3.4, 8, 2.6, 0.5, 2.7, 9.1, 11.6, 4, 1.7, 1.9, 12.9, 4.8, 6.6, 3.1, 7.8, 13, 10.2, 7.5, 7.7, 11.6, 10.9.

A summary of the key descriptive statistics for the insurance data is provided in table 2 where we can observe that the data suggesting a slightly right-skewed distribution, and the negative kurtosis value reveals a platykurtic distribution.

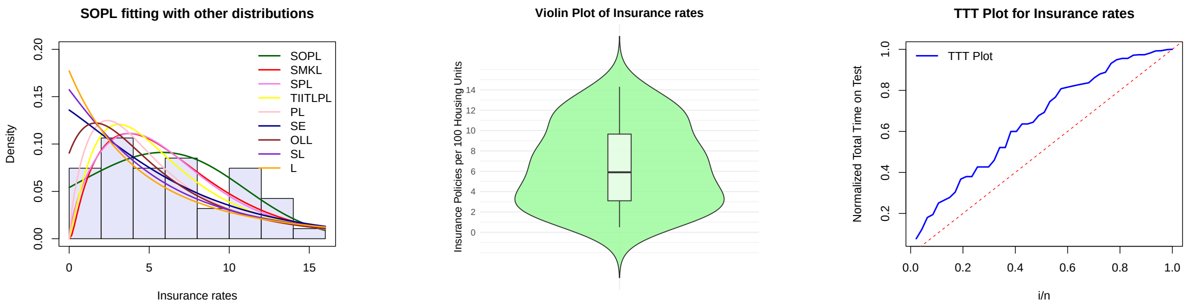


Figure 5: PDFs, Violin and TTT plots for the Insurance data.

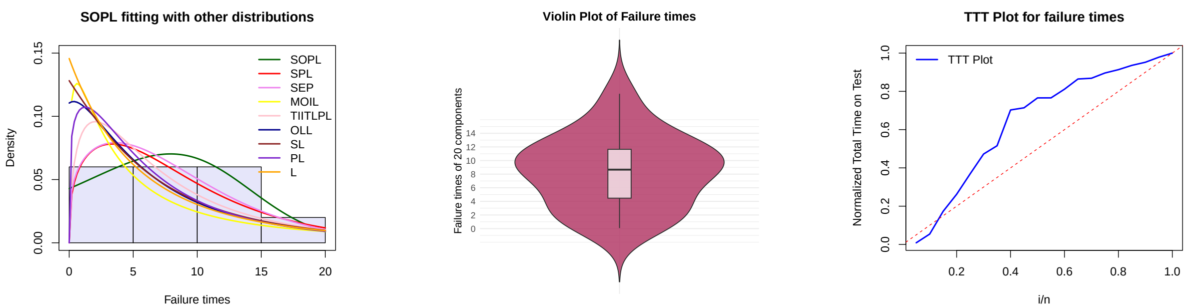


Figure 6: PDFs, Violin and TTT plots for the failure data.

Figure 5 provides a visual assessment of the SOPL distribution. The density plots show how well the SOPL distribution fits with the insurance data against other models. The violin plot illustrates the overall distribution. Meanwhile, the TTT plot reveals a decreasing hazard pattern.

Table 3 presents the goodness of fit measures and information criteria for various models fitted to the insurance data. These results suggest that the SOPL distribution offers a better fit compared to other considered models for this dataset.

The P-P and Q-Q plots in figure 7 show that the SOPL model closely matches the data, compared to the competing models, highlighting its better fit to the insurance data.

Dataset 2: The second data set refers to the failure of 20 components [32]. The data are 0.072, 4.763, 8.663, 12.089, 0.477, 5.284, 9.511, 13.036, 1.592, 7.709, 10.636, 13.949, 2.475, 7.867, 10.729, 16.169, 3.597, 8.661, 11.501, 19.809. The competitive models considered are Type-II Topp-Leone power Lomax (TIITLPL) [31], Sine Power Lomax (SPL) [30], Sine exponential pareto (SEP) [33], Marshall-olkin Inverse Lomax (MOIL) [34], Odd Lomax - Lomax (OLL) [25], Sine Lomax (SL) [16], Power Lomax (PL) [26], Lomax (L) [20].

Table 4 presents the key descriptive statistics for the failure data and figure 6 includes estimated pdf plots for the proposed model along with the competitive models, violin plot of the failure data, and TTT plot showing increasing decreasing hazard pattern.

Goodness of fit measures and information criteria for various models fitted to the failure data are given in table 5, results in better fit of the SOPL model over other competitive models.

Better alignment of the theoretical probabilities with the empirical probabilities for the SPOL model over the competitive models is shown through the P-P plots in figure 8, where Q-Q plots are also presented.

8. Conclusion

This study presented a new flexible family of distributions named the Sine Odd Perks-G (SOP-G) family, with a specific focus on the Lomax based sub model which is named the Sine Odd-Perks Lomax (SOPL) distribution. The theoretical characteristics of the model were thoroughly investigated, including its density shapes, hazard rate behaviors, and higher-order moment structures. The SOPL distribution demonstrated a remarkable ability to model diverse data patterns, exhibiting right-skewed, left-skewed, symmetric, and n-shaped density forms, along with various hazard shapes such as increasing, decreasing, inverted bathtub, and sigmoid patterns. Parameter estimation was successfully performed using five different estimation methods, the simulation study demonstrates that the least squares method exhibits superior performance among other methods in decreasing bias and mse values of the estimators as the sample size increased. Graphical diagnostics including violin plots, total time on test (TTT) plots, and P–P and Q-Q plots, along with goodness-of-fit comparisons, validated the model's strong adaptability and superiority over several existing models when applied to real life datasets involving actuarial related insurance data and failure data. As a part of the future scope, exploring the SOP-G family with different baseline distributions could lead to additional flexible models suitable for a wider range of applications in survival analysis, reliability engineering, and financial risk assessment. However inappropriate selection of the baseline distribution leads to reduced performance of the SOP-G family. Furthermore, the presence of multiple parameters may increase the complexity of the estimation procedure, which fall under the limitations of the proposed model.

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9. Appendix

Table 1: Moments of the SOPL Distribution.

Parameters	a	μ'_1	μ'_2	μ'_3	μ'_4	Skewness	Kurtosis	Variance
$\beta = 2.7, \lambda = 0.8, \delta = 0.5$	1	0.9940	1.8996	5.5590	22.9289	4.5609	11.0222	0.9116
	3	0.7558	1.1920	3.0568	11.5568	6.1987	14.0708	0.6207
	5	0.7020	1.0515	2.6052	9.6362	6.7204	15.0605	0.5587
	7	0.6781	0.9915	2.4178	8.8540	6.9778	15.5520	0.5317
	9	0.6646	0.9583	2.3154	8.4304	7.1313	15.8459	0.5167
$\beta = 1.6, \lambda = 2.8, \delta = 0.7$	1	0.3150	0.1786	0.1454	0.1563	3.0661	7.9049	0.0794
	3	0.2425	0.1148	0.0824	0.0813	4.2979	10.0332	0.0560
	5	0.2260	0.1019	0.0708	0.0683	4.6909	10.7240	0.0508
	7	0.2186	0.0963	0.0659	0.0630	4.8849	11.0669	0.0485
	9	0.2144	0.0932	0.0633	0.0601	5.0005	11.2719	0.0473
$\beta = 3.4, \lambda = 1.9, \delta = 0.4$	2	0.3611	0.2796	0.3662	0.7435	7.4605	17.1634	0.1493
	4	0.3169	0.2239	0.2758	0.5390	8.5191	19.3000	0.1234
	6	0.3014	0.2056	0.2474	0.4767	8.9631	20.2059	0.1147
	8	0.2935	0.1964	0.2335	0.4468	9.2076	20.7071	0.1103
	10	0.2887	0.1910	0.2254	0.4292	9.3625	21.0254	0.1076
$\beta = 4.6, \lambda = 2.5, \delta = 0.3$	2	0.2860	0.1872	0.2250	0.4640	10.5536	25.0600	0.1054
	4	0.2499	0.1486	0.1675	0.3329	11.9701	28.2484	0.0861
	6	0.2373	0.1359	0.1497	0.2933	12.5644	29.6023	0.0796
	8	0.2309	0.1297	0.1410	0.2743	12.8919	30.3520	0.0764
	10	0.2270	0.1260	0.1359	0.2632	13.0993	30.8283	0.0744

Table 2: Descriptive Statistics of Insurance data

Minimum	Maximum	Mean	Median	Mode	Variance	Skewness	Kurtosis
0.0720	19.8090	8.4295	8.6620	3.0000	28.3243	0.1770	-0.5691

Table 3: Adequacy measures of models for Insurance data

Models	W*	A*	Dn	p-value	AIC	CAIC	BIC	HQIC
SOPL	0.0555	0.3748	0.0875	0.8642	262.3089	263.2613	269.7095	265.0938
SMKL	0.0789	0.5189	0.1018	0.7142	263.2392	263.7973	268.7896	265.3279
SPL	0.0852	0.5579	0.1147	0.5666	264.1932	264.7513	269.7436	266.2818
TIITLPL	0.1131	0.7233	0.1308	0.3974	268.7913	269.7437	276.1919	271.5762
PL	0.1514	0.9486	0.1323	0.3832	273.4039	273.9621	278.9544	275.4926
SE	0.0806	0.5271	0.1471	0.2612	269.7437	269.6029	271.3641	270.2102
OLL	0.1219	0.7747	0.1608	0.1760	275.6988	276.2570	281.2493	277.7875
SL	0.1030	0.6634	0.1675	0.1429	278.0781	278.3508	281.7784	279.4705
L	0.1131	0.7246	0.1814	0.0908	282.6262	282.8990	286.3265	284.0187

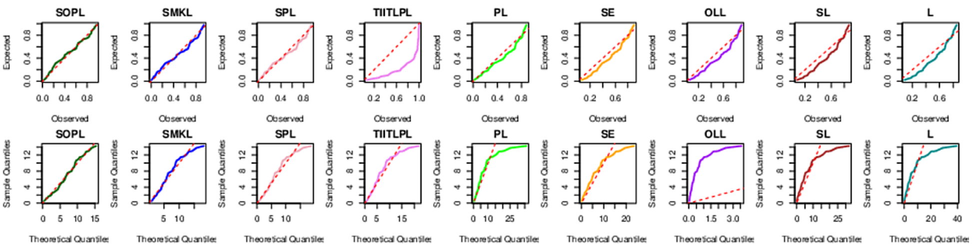

Figure 7: P-P ad Q-Q plots of SOPL along with comparitive models for Insurance data.

Table 4: Descriptive Statistics of failures data

Minimum	Maximum	Mean	Median	Mode	Variance	Skewness	Kurtosis
0.5000	14.3000	6.5298	5.9000	3.0000	15.7330	0.2804	-1.0991

Table 5: Adequacy measures of models for failures data

Models	W*	A*	Dn	p-value	AIC	CAIC	BIC	HQIC
SOPL	0.0477	0.2720	0.1049	0.9640	126.7842	129.4509	130.7672	127.5617
SPL	0.1430	0.8303	0.1821	0.4666	130.3979	131.8979	133.3851	130.9810
SEP	0.1152	0.6667	0.1969	0.3710	128.5670	130.0670	131.5542	129.1501
MOIL	0.2897	1.6574	0.2252	0.2258	141.9039	143.4039	144.8911	142.4870
TIITLPL	0.1949	1.1308	0.2400	0.1690	135.6512	138.3179	139.6341	136.4287
OLL	0.1956	1.1317	0.2580	0.1159	134.5317	136.0317	137.5189	135.1148
SL	0.1793	1.0408	0.2653	0.0987	131.5037	132.2096	133.4951	131.8924
PL	0.2015	1.1666	0.2708	0.0871	134.4074	135.9074	137.3946	134.9905
L	0.1955	1.1344	0.2705	0.0878	132.9880	133.6939	134.9795	133.3767

Figure 8: P-P ad Q-Q plots of SOPL along with comparitive models for failure data

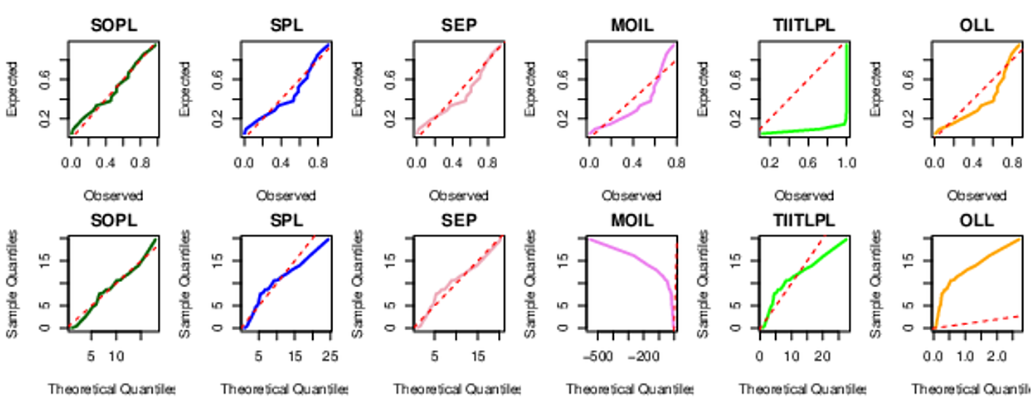

Table 6: Simulation results of different estimation methods across sample sizes for Set-I.

Set-I: $\alpha = 0.1, \beta = 0.2, \lambda = 0.6, \delta = 0.8$

Sample size→		20			50			70			100			200			350			500		
Method	Parameters	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
MLE	α	0.2244	0.1244	0.6816	0.2109	0.1109	1.1261	0.1756	0.0756	0.0826	0.1469	0.0469	0.0430	0.1573	0.0573	0.0472	0.1439	0.0439	0.0301	0.1378	0.0378	0.0226
	β	0.2397	0.0397	0.1332	0.2594	0.0594	0.1452	0.2843	0.0843	0.1873	0.2979	0.0979	0.1895	0.2728	0.0728	0.1061	0.2424	0.0424	0.0593	0.2384	0.0384	0.0405
	λ	1.0313	0.4313	9.6601	0.9587	0.3587	3.1287	0.8932	0.2932	1.3727	0.9485	0.3485	1.8526	0.8118	0.2118	0.6177	0.8612	0.2612	0.7241	0.7679	0.1679	0.3548
	δ	0.8654	0.0654	0.0671	0.8395	0.0395	0.0620	0.8370	0.0370	0.0564	0.8114	0.0114	0.0459	0.8274	0.0274	0.0478	0.8251	0.0251	0.0392	0.8219	0.0219	0.0330
MPS	α	0.1873	0.0873	0.2287	0.1400	0.0400	0.1496	0.2192	0.1192	9.1776	0.1285	0.0285	0.0401	0.1097	0.0097	0.0127	0.1151	0.0151	0.0386	0.1116	0.0116	0.0185
	β	0.3852	0.1852	0.8601	0.3336	0.1336	0.2264	0.3451	0.1451	0.2809	0.3644	0.1644	0.3306	0.3267	0.1267	0.2021	0.3054	0.1054	0.0979	0.2805	0.0805	0.0674
	λ	3.1060	2.5060	1285.04	1.3151	0.7151	7.6590	1.3780	0.7780	6.7635	1.3605	0.7605	8.6195	1.1538	0.5538	2.6249	0.9766	0.3766	0.8435	0.9862	0.3862	1.0406
	δ	0.7632	-0.0368	0.0437	0.7614	-0.0386	0.0433	0.7523	-0.0477	0.0441	0.7554	-0.0446	0.0481	0.7502	-0.0498	0.0384	0.7513	-0.0487	0.0372	0.7626	-0.0374	0.0315
LSE	α	0.1204	0.0204	0.0433	0.1142	0.0142	0.0057	0.1065	0.0065	0.0020	0.1082	0.0082	0.0020	0.1070	0.0070	0.0014	0.1053	0.0053	0.0011	0.1032	0.0032	0.0007
	β	0.2078	0.0078	0.0154	0.2034	0.0034	0.0091	0.2050	0.0050	0.0086	0.1972	-0.0028	0.0048	0.1958	-0.0042	0.0039	0.1993	-0.0007	0.0036	0.2008	0.0008	0.0032
	λ	0.7196	0.1196	0.2532	0.6555	0.0555	0.0765	0.6709	0.0709	0.0985	0.6679	0.0679	0.0806	0.6438	0.0438	0.0286	0.6250	0.0250	0.0222	0.6249	0.0249	0.0162
	δ	0.7985	-0.0015	0.0110	0.8061	0.0061	0.0071	0.8001	0.0001	0.0057	0.8040	0.0040	0.0048	0.8059	0.0059	0.0036	0.8062	0.0062	0.0035	0.8034	0.0034	0.0030
CvM	α	0.1234	0.0234	0.0281	0.1125	0.0125	0.0056	0.1109	0.0109	0.0033	0.1067	0.0067	0.0020	0.1058	0.0058	0.0011	0.1039	0.0039	0.0008	0.1061	0.0061	0.0009
	β	0.2037	0.0037	0.0185	0.2060	0.0060	0.0138	0.1996	-0.0004	0.0071	0.2022	0.0022	0.0106	0.1980	-0.0020	0.0049	0.1995	-0.0005	0.0032	0.1967	-0.0033	0.0032
	λ	0.6923	0.0923	0.3000	0.6569	0.0569	0.0838	0.6491	0.0491	0.0594	0.6555	0.0555	0.0601	0.6346	0.0346	0.0317	0.6278	0.0278	0.0208	0.6295	0.0295	0.0231
	δ	0.8206	0.0206	0.0134	0.8103	0.0103	0.0075	0.8108	0.0108	0.0066	0.8089	0.0089	0.0070	0.8079	0.0079	0.0040	0.8051	0.0051	0.0034	0.8075	0.0075	0.0032
ADE	α	0.1333	0.0333	0.0855	0.1124	0.0124	0.0080	0.1067	0.0067	0.0028	0.1057	0.0057	0.0029	0.1044	0.0044	0.0020	0.1032	0.0032	0.0013	0.1045	0.0045	0.0010
	β	0.2144	0.0144	0.0172	0.2061	0.0061	0.0099	0.2112	0.0112	0.0124	0.2131	0.0131	0.0115	0.2106	0.0106	0.0096	0.2111	0.0111	0.0083	0.2033	0.0033	0.0043
	λ	0.7525	0.1525	0.8993	0.6817	0.0817	0.1734	0.6630	0.0630	0.0974	0.6592	0.0592	0.0737	0.6644	0.0644	0.0628	0.6476	0.0476	0.0487	0.6306	0.0306	0.0283
	δ	0.8073	0.0073	0.0135	0.8074	0.0074	0.0096	0.8018	0.0018	0.0073	0.7981	-0.0019	0.0070	0.7969	-0.0031	0.0066	0.7965	-0.0035	0.0055	0.8025	0.0025	0.0040

Table 7: Simulation results of different estimation methods across sample sizes for Set-II.

Set-II: $\alpha = 0.05, \beta = 1.2, \lambda = 0.6, \delta = 0.8$

Sample size→		20			50			70			100			200			350			500		
Method	Parameters	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
MLE	α	0.1358	0.0858	4.3407	0.0716	0.0216	0.0117	0.0690	0.0190	0.0071	0.0682	0.0182	0.0072	0.0700	0.0200	0.0091	0.0645	0.0145	0.0040	0.0595	0.0095	0.0021
	β	1.5381	0.3381	3.6021	1.6145	0.4145	2.2893	1.6299	0.4299	2.7131	1.5373	0.3373	1.4759	1.4300	0.2300	1.0081	1.3530	0.1530	0.5465	1.3096	0.1096	0.3454
	λ	0.7602	0.1602	0.9460	0.8319	0.2319	1.7561	0.7946	0.1946	0.5648	0.7677	0.1677	0.4229	0.7401	0.1401	0.2942	0.6878	0.0878	0.1812	0.6723	0.0723	0.1109
	δ	0.9889	0.1889	0.8154	0.9079	0.1079	0.3228	0.8745	0.0745	0.2342	0.8719	0.0719	0.1817	0.8723	0.0723	0.1773	0.8641	0.0641	0.1414	0.8394	0.0394	0.0730
MPSE	α	0.1024	0.0524	0.3070	0.0778	0.0278	0.0165	0.0679	0.0179	0.0076	0.0659	0.0159	0.0058	0.0623	0.0123	0.0075	0.0564	0.0064	0.0023	0.0550	0.0050	0.0014
	β	1.6231	0.4231	4.8129	1.6786	0.4786	2.7007	1.6954	0.4954	2.8887	1.7298	0.5298	2.9206	1.5375	0.3375	0.9936	1.4308	0.2308	0.6658	1.3594	0.1594	0.4762
	λ	0.9371	0.3371	2.6701	0.9053	0.3053	1.4576	0.9064	0.3064	0.9588	0.9251	0.3251	1.2105	0.8340	0.2340	0.4504	0.7597	0.1597	0.2610	0.7299	0.1299	0.1691
	δ	0.9308	0.1308	0.8265	0.8777	0.0777	0.3266	0.8474	0.0474	0.2577	0.8156	0.0156	0.1715	0.8088	0.0088	0.1481	0.7958	-0.0042	0.0781	0.8003	0.0003	0.0595
LSE	α	0.0646	0.0146	0.0053	0.0565	0.0065	0.0013	0.0546	0.0046	0.0011	0.0524	0.0024	0.0007	0.0511	0.0011	0.0005	0.0508	0.0008	0.0003	0.0510	0.0010	0.0004
	β	1.2187	0.0187	0.2636	1.2416	0.0416	0.1995	1.2440	0.0440	0.2102	1.2649	0.0649	0.1722	1.2768	0.0768	0.1644	1.2642	0.0642	0.1235	1.2543	0.0543	0.1270
	λ	0.6731	0.0731	0.2732	0.6306	0.0306	0.0413	0.6371	0.0371	0.0523	0.6358	0.0358	0.0439	0.6453	0.0453	0.0456	0.6385	0.0385	0.0402	0.6460	0.0460	0.0393
	δ	0.8189	0.0189	0.0396	0.8123	0.0123	0.0241	0.8118	0.0118	0.0241	0.8031	0.0031	0.0189	0.7941	-0.0059	0.0185	0.7973	-0.0027	0.0165	0.7986	-0.0014	0.0190
CVE	α	0.0591	0.0091	0.0037	0.0554	0.0054	0.0014	0.0526	0.0026	0.0008	0.0527	0.0027	0.0007	0.0514	0.0014	0.0004	0.0508	0.0008	0.0004	0.0508	0.0008	0.0003
	β	1.2518	0.0518	0.3724	1.2347	0.0347	0.2407	1.2471	0.0471	0.1835	1.2467	0.0467	0.1513	1.2565	0.0565	0.1537	1.2474	0.0474	0.1166	1.2513	0.0513	0.1116
	λ	0.6551	0.0551	0.0830	0.6336	0.0336	0.0566	0.6488	0.0488	0.0707	0.6262	0.0262	0.0304	0.6324	0.0324	0.0319	0.6464	0.0464	0.0423	0.6315	0.0315	0.0285
	δ	0.8271	0.0271	0.0418	0.8243	0.0243	0.0312	0.8073	0.0073	0.0220	0.8107	0.0107	0.0209	0.8046	0.0046	0.0186	0.8011	0.0011	0.0192	0.8014	0.0014	0.0163
ADE	α	0.0605	0.0105	0.0038	0.0557	0.0057	0.0016	0.0552	0.0052	0.0015	0.0515	0.0015	0.0011	0.0529	0.0029	0.0009	0.0517	0.0017	0.0006	0.0517	0.0017	0.0005
	β	1.2839	0.0839	0.4213	1.3057	0.1057	0.4628	1.2873	0.0873	0.2890	1.3403	0.1403	0.3757	1.3093	0.1093	0.3006	1.2897	0.0897	0.1992	1.2814	0.0814	0.1900
	λ	0.6730	0.0730	0.1576	0.6574	0.0574	0.0827	0.6516	0.0516	0.0846	0.7044	0.1044	0.1439	0.6606	0.0606	0.0745	0.6546	0.0546	0.0590	0.6605	0.0605	0.0652
	δ	0.8217	0.0217	0.0488	0.8126	0.0126	0.0381	0.8162	0.0162	0.0384	0.7908	-0.0092	0.0353	0.8029	0.0029	0.0336	0.7992	-0.0008	0.0261	0.7992	-0.0008	0.0267

Table 8: Simulation results of different estimation methods across sample sizes for Set-III.

Set-III: $\alpha = 0.02, \beta = 1.2, \lambda = 2.6, \delta = 1.8$

Sample size→		20			50			70			100			200			350			500		
Method	Parameters	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
MLE	α	0.02936	0.00936	0.00383	0.02902	0.00902	0.00235	0.02383	0.00383	0.00089	0.02395	0.00395	0.00114	0.02425	0.00425	0.00047	0.02198	0.00198	0.00024	0.02211	0.00211	0.00018
	β	1.53213	0.33213	3.34468	1.51216	0.31216	1.78331	1.63987	0.43987	2.27042	1.64170	0.44170	2.11068	1.44543	0.24543	1.53079	1.44341	0.24341	0.76033	1.34807	0.14807	0.48544
	λ	2.87907	0.27907	2.53106	2.98120	0.38120	2.64583	3.09131	0.49131	3.17312	3.00190	0.40190	2.40175	2.85050	0.25050	1.53079	2.73166	0.13166	0.76033	2.69077	0.09077	0.52515
	δ	2.05285	0.25285	2.46699	1.93933	0.13933	1.05339	1.80639	0.00639	0.62791	1.83211	0.03211	0.82515	1.89889	0.09889	0.60489	1.85993	0.05993	0.47048	1.85149	0.05149	0.28216
MPSE	α	0.03731	0.01731	0.00386	0.03066	0.01066	0.00233	0.03003	0.01003	0.00185	0.02523	0.00523	0.00082	0.02368	0.00368	0.00042	0.02287	0.00287	0.00034	0.02115	0.00115	0.00014
	β	1.71289	0.51289	4.52839	1.75887	0.55887	2.83587	1.66190	0.46190	2.39354	1.74287	0.54287	2.36070	1.56970	0.36970	1.36155	1.48466	0.28466	0.86619	1.43928	0.23928	0.60569
	λ	2.99367	0.39367	5.67451	3.14408	0.54408	5.10675	3.13320	0.53320	3.92583	3.18543	0.58543	3.98246	2.91809	0.31809	1.58631	2.77583	0.17583	0.76050	2.75535	0.15535	0.59829
	δ	2.06472	0.26472	2.68052	1.92920	0.12920	1.78622	1.88049	0.08049	1.22829	1.80170	0.00170	0.90882	1.81134	0.01134	0.56008	1.81158	0.01158	0.48996	1.77460	-0.02540	0.25679
LSE	α	0.02387	0.00387	0.00063	0.02249	0.00249	0.00025	0.02076	0.00076	0.00007	0.02092	0.00092	0.00007	0.02075	0.00075	0.00006	0.02019	0.00019	0.00004	0.01968	-0.00032	0.00003
	β	1.30481	0.10481	0.42175	1.27322	0.07322	0.27402	1.28493	0.08493	0.21010	1.25386	0.05386	0.15705	1.26622	0.06622	0.12315	1.27441	0.07441	0.14107	1.29589	0.09589	0.15320
	λ	2.67961	0.07961	0.43050	2.66753	0.06753	0.47755	2.67143	0.07143	0.27015	2.67765	0.07765	0.26858	2.64503	0.04503	0.18945	2.64756	0.04756	0.15102	2.71118	0.11118	0.23841
	δ	1.81101	0.01101	0.17193	1.82208	0.02208	0.13274	1.79171	-0.00829	0.08774	1.80192	0.00192	0.10246	1.79635	-0.00365	0.08084	1.79138	-0.00862	0.07851	1.76388	-0.03612	0.08871
CVE	α	0.02346	0.00346	0.00059	0.02040	0.00040	0.00007	0.02060	0.00060	0.00009	0.01985	-0.00015	0.00006	0.01985	-0.00015	0.00005	0.01999	-0.00001	0.00004	0.01991	-0.00009	0.00004
	β	1.27213	0.07213	0.35603	1.27099	0.07099	0.20197	1.28391	0.08391	0.27504	1.28997	0.08997	0.20797	1.28507	0.08507	0.18270	1.27657	0.07657	0.14448	1.27801	0.07801	0.13600
	λ	2.68808	0.08808	0.64326	2.67191	0.07191	0.42547	2.70722	0.10722	0.37453	2.68702	0.08702	0.27926	2.68743	0.08743	0.23751	2.68001	0.08001	0.24994	2.67732	0.07732	0.17197
	δ	1.84597	0.04597	0.17858	1.81998	0.01998	0.12086	1.78804	-0.01196	0.09863	1.78906	-0.01094	0.10178	1.78419	-0.01581	0.09023	1.78859	-0.01141	0.08935	1.78001	-0.01999	0.07653
ADE	α	0.02495	0.00495	0.00171	0.02177	0.00177	0.00022	0.02102	0.00102	0.00015	0.02065	0.00065	0.00010	0.02006	0.00006	0.00009	0.02050	0.00050	0.00007	0.02053	0.00053	0.00007
	β	1.29569	0.09569	0.50156	1.32168	0.12168	0.43952	1.32545	0.12545	0.35535	1.29991	0.09991	0.26268	1.34451	0.14451	0.28878	1.71656	0.10957	0.30100	1.29339	0.09339	0.21423
	λ	2.76265	0.16265	0.75028	2.75578	0.15578	0.72043	2.78527	0.18527	0.68650	2.74657	0.14657	0.45994	2.73493	0.13493	0.37724	2.71683	0.11683	0.30100	2.68103	0.08103	0.25812
	δ	1.83853	0.03853	0.22916	1.80379	0.00379	0.17961	1.78277	-0.01723	0.17855	1.78520	-0.01480	0.13646	1.76833	-0.03167	0.14463	1.78344	-0.01656	0.12992	1.79811	-0.00189	0.12518

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