**IDEALS & GENERALIZED DERIVATIONS IN  
 PRIME RINGS***M. Rais Khan, Deepa Arora & M. Ali Khan*

**ABSTRACT**

Let R be a prime ring and F and G be generalized derivations of R with associated derivations d and g respectively. In the present paper, we shall investigate the commutativity of R admitting generalized derivations F and G satisfying any one of the properties:   
(i) F(x)x = x G(x), (ii) F(x2) = x2 , (iii) [F(x), y] = [x, G(y)], (iv) d(x)F(y) = xy, (v) F([x, y]) = [F(x), y] + [d(y), x] and (vi) F(x ◦ y) = F(x) ◦ y − d(y) ◦ x for all x, y in some appropriate subset of R.

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 **1. INTRODUCTION**

Throughout this paper, R will represent an associative ring with center Z(R). Recall that a ring R is prime if aRb = 0 implies a = 0 or b = 0. If R has an involution σ, then R is said to be prime if aRb = aRσ(b) = 0 implies a = 0 or b = 0. Every prime ring equipped with an involution is prime but the converse need not be true in general. As an example, taking S = R× R0 where R0 is an opposite ring of a prime ring R with (x, y) = (y, x). Then S is not prime if (0, a) S (a, 0) = 0. But if we take (a, b) S (x, y) = 0 and (a, b) S σ ((x, y)) = 0, then aRx × yRb = 0 and aRy × xRb = 0, and thus aRx = yRb = aRy = xRb = 0. This shows that R is prime (see for reference [9]). An ideal I of R is a ideal if I is invariant under σ i.e. σ (I) = I. Oukhtite et al. defined a set of symmetric and skew symmetric elements of R as Saσ(R) = {x ∈R | σ(x) = x}. For any x, y R, the symbol [x, y] stands for commutator xy − yx and the symbol x ◦ y denotes the anti-commutator xy + yx. We shall make extensive use of the following basic commutator identities: [xy, z] = x[y, z] + [x, z]y, [x, yz] = y[x, z]+[x, y]z, xo(yz) = (xoy)z−y[x, z] = y(xoz)+[x, y]z and (xy)oz = x(yoz)− [x, z]y = (xoz)y + x[y, z].

As defined by Bresar [6], an additive map F: R → R is called a generalized derivation associated with d if there exists a derivation d: R → R (an additive map d: R → R is called a derivation if d (xy) = d(x)y + xd(y) holds for all x, y ∈ R) such that F (xy) = F(x) y + xd(y) for all x, y ∈ R. One can easily check that the notion of generalized derivation covers the notions of a derivation and a left multiplier (i.e. F (xy) = F(x) y for all x, y ∈ R). Particularly, we can observe that: For a fixed a ∈ R, the map da : R → R defined by da(x) = [a, x] for all x ∈ R is a derivation which is said to be an inner derivation. An additive map ga,b: R → R is called a generalized inner derivation if ga,b(x) = ax + xb for some fixed a, b ∈ R. It is easy to see that if ga,b(x) is a generalized inner derivation, then

ga,b(x) (xy) = ga,b(x)y + xd−b(y) for all x, y ∈ R, where d−b is an inner derivation.

Several authors [1, 2, 3, 17, 18, 19, 20] have established numerous results concerning derivations and generalized derivations of prime rings. In 2005, Oukhtite et al. conferred an extension of prime rings in the form of -prime rings and proved a number of results which hold true for prime rings (see for references [9 - 16]). In [7] and [8] author et al. extended results concerning derivations and generalized derivations of prime rings to some more general settings. Ashraf et al. too contributed to this newly emerged theory in [5], apart from great deal of work in the field of prime rings.

Recently, in [4], Ashraf et al. extended some known theorems for derivations to generalized derivations in the setting of semiprime rings.

Now a natural question arises: Under what additional condition these results can be extended in semiprime rings. However, in this perspective, we prove the results for prime rings exhibiting generalized derivations F and G associated with derivations d and g respectively and hope for similar conversion to semiprime rings in near future.

We define the following properties for all x, y I, ideal of prime ring R, such that:

(P1) F(x) x x G(x) = 0.

(P2) F(x2) x2 = 0.

(P3) [F(x), y] [x, G(y)] = 0.

(P4) d(x) F(y) xy = 0.

(P5) F([x, y]) = [F(x), y] + [d(y), x].

(P6) F(x) ◦ y − d(y) ◦ x = 0.

**2. MAIN RESULTS**

To prove our results, we need the following known lemmas:

**Lemma 2.1** ([10, Lemma 3.1]) *Let R be a σ-prime ring and let I be a nonzero σ−ideal of R. If a, b in R satisfy a I b = a I σ(b) = 0, then a = 0 or b = 0.*

**Lemma 2.2** ([11, Lemma 2.2]) *Let I be a nonzero σ-ideal of R and 0 d be a derivation on R which commutes with σ. If [x, R] I d(x) = 0 for all x ∈ I, then R is commutative.*

We begin with

**Theorem 2.1** *Let R be a two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g which commutes with respectively. Let R satisfy the property (P1).Then R is commutative.*

**Proof of Theorem 2.1**: (i) By our hypothesis (P1), we have

F(x) x = x G(x) for all x ∈ I. (1)

Linearizing (1) we find that

F(x) y + F(y) x = xG(y) + yG(x) for all x, y ∈ I. (2)

Replace x by xy in (2), to get

F(x) y2 + x d(y) y + F(y) xy = xy G(y) + y G(x) y + yx g(y) for all x, y ∈ I. (3)

Multiplying relation (2) with y from right yields

F(x) y2 + F(y) xy = x G(y) y + y G(x) y for all x, y ∈ I. (4)

Combining (3) and (4), we obtain

x d(y) y = yx g(y) + x [y, G(y)] for all x, y ∈ I. (5)

For any r ∈ R, replacing x by rx in (5), we get

rx d(y)y = yrx g(y) + rx [y, G(y)] for all x, y ∈ I, r ∈ R. (6)

On left multiplying (5) by r, we have

rx d(y) y = ryx g(y) + rx [y, G(y)] for all x, y ∈ I, r ∈ R. (7)

From (6) and (7), we get [y, r] x g(y) = 0. Therefore,

[y, r] I g(y) = 0 for all y ∈ I, r ∈ R. (8)

Let y I ∩ Saσ(R); since g commutes with σ, the relation (8) yields

[y, r] I g(y) = 0 = [y, r] I σ (g(y)) for all y ∈ I, r ∈ R. (9)

By virtue of Lemma 2.1, either [y, r] = 0 or g(y) = 0.

Let y ∈ I. As y + (y) is an element of I, then

[y + (y), r] = 0 or g(y + (y)) = 0 for all r ∈ R.

Case 1: If [y + (y), r] = 0, the fact that y (y) I, yields

[y (y), r] = 0 or g(y - (y)) = 0 r ∈ R.

If [y (y), r] = 0, then 0 = [y - (y), r] + [y + (y), r] = 2[y, r] = 0.

Therefore, [y, r] = 0, since cha*R*  2.

If g(y - (y)) = 0 r ∈ R, then g(y) = g ( (y)) = (g(y)).

Hence, by application of Lemma 2.1 equation (8) implies [y, r] = 0 or g(y) = 0.

Case 2: If g(y + (y)) = 0, then g(y) = g ( (y)) = (g(y)), and in view of (8)

[y, r] I g(y) = 0 = [y, r] I σ (g(y)).

Hence by Lemma 2.1, we arrive at [y, r] = 0 or g(y) = 0.

If g(y) = 0, then for any r in R, we find that y d(r) = 0 for all y ∈ I.

Hence, I d(r) = I R d(r) = (I) R d(r) = 0.

Since I 0 and R is - prime, we obtain d(R) = 0. Hence, d = 0 a contradiction.

Now suppose that [y, r] = 0. Then for any s in R, we can write

0 = [sy, r] = [s, r] y = [s, r] I = [s, r] R I = [s, r] R (I) = 0.

Since I 0, in view of - primness of R, we obtain [s, r] = 0 for all r, s ∈ R.

Hence R is commutative.

(ii) Similarly we can prove that R is commutative, if F(x) x + x G(x) = 0 for all x ∈ I.

Following on the same lines with necessary variations and taking G = F or G = −F in Theorem 2.1, we get the following corollary:

**Corollary 2.1** *Let R be a two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d which commutes with such that [F(x), x] = 0 for all x I or if F(x) ◦ x = 0 for all x I, then R is commutative.*

**Theorem 2.2** *Let R be a two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits generalized derivations F with associated nonzero derivation d which commutes with such that the property (P2) is satisfied. Then R is commutative.*

**Proof of Theorem 2.2**: (i) From the hypothesis of (P2), we write

F(x2) = x2 for all x I.

Replacing x by x + y in the above relation, we get

F(x2 + y2 + xy + yx) = x2 + y2 + xy + yx for all x, y I. (10)

Using (P2) in (10), we obtain

F(xy + yx) = xy + yx for all x, y I. (11)

Rewriting equation (11), we get F(x ◦ y) = x ◦ y for all x, y I.

Hence, by virtue of result [14, Theorem 2.2], we get the required result.

(ii) Further, if F(x2) + x2 = 0 for all x I, then we find that F(xoy) + (xoy) = 0 for all x, y I. Following the same technique as used in the proof of [14, Theorem 2.2], we get the required result.

**Theorem 2.3** *Let R be a two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g which commutes with respectively. If R satisfies property (P3), then R is commutative.*

**Proof of Theorem 2.3**: (i) By the hypothesis of (P3), we have

[F(x), y] = [x, G(y)] for all x, y I.

Replacing y by yx in the above expression, we obtain

y [F(x), x] = [x, y] g(x) + y[x, g(x)] for all x, y I. (12)

For any r R, again replacing y by ry in (12) and applying (12), we get

[x, r] y g(x) = 0 for all x, y I.

Therefore, [x, R] I g(x) = 0 for all x I.

Hence, by application of Lemma 2.2 we conclude that R is commutative.

(ii) Using the same techniques as used above we conclude R is commutative, if [F(x), y] + [x, G(y)] = 0 for all x, y I.

Substituting G = F or G = −F in Theorem 2.3 and applying similar arguments, we can prove the following corollary.

**Corollary 2.3:** *Let R be two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits generalized derivations F and G with associated nonzero derivations d and g which commutes with respectively. If [F(x), y] = [x, F(y)] for all x, y I or if [F(x), y] + [x, F(y)] = 0 for all x, y I, then R is commutative.*

**Theorem 2.4** *Let R be a two torsion free prime ring and I a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with . Let R satisfy property (P4).Then R is commutative.*

**Proof of Theorem 2.4**: (i) From the hypothesis of (P4), we have

d(x)F(y) = xy for all x, y I. (13)

Replacing y by yx in (13) and using (P4), we get

d(x) y d(x) = 0 for all x, y I. (14)

This implies that d(x) xy d(x) = 0 for all x, y I. (15)

Equation (14) also yields that

x d(x) y d(x) = 0 for all x, y I. (16) Equation (15) together with equation (16) gives

[d(x), x] y d(x) = 0 for all x, y I. Therefore [d(x), x] I d(x) = 0 for all x I.

Since d commutes with σ, Lemma 2.1 gives [d(x), x] = 0 or d(x) = 0 x I.

If d(x) = 0 x I, then for any r ∈ R, we have

0 = d (xr) = x d(r) = I d(r) = I R d(r) = (I) R d(r) = 0.

Since I is nonzero, -primeness of R yields d = 0 for all r in R, a contradiction.

Next, suppose that [d(x), x] = 0 x I.

Then, by the application of techniques used in the proof of the result ([11, Theorem 1.2]), we conclude that R is commutative.

(ii) If d(x) F(y) + xy = 0 for all x, y I, then using the same techniques as used above with necessary variations we get the required result.

**Theorem 2.5** Let R be a 2-torsion free - prime ring and I be a nonzero -ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with such that property (P5) is satisfied. Then R is commutative.

**Proof of Theorem 2.5:** By our hypothesis (P5), we have

F([x, y]) = [F(x), y] + [d(y), x]. (17)

Replacing y by yx in (17) and employing (17), we find that

2[x, y] d(x) = y [F(x), x] + y [d(x), x] for all x, y I. (18)

For any r R, again replacing y by ry in (18) and applying (18), we get

2[x, r] y d(x) = 0 for all x, y I.

Since R is 2-torsion free, we get [x, r] y d(x) = 0 for all x, y I and r R.

Therefore, [x, R] I d(x) = 0 for all x I and r R.

By application of Lemma 2.2, we conclude that R is commutative.

**Theorem 2.6** *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation d commuting with. Let R satisfies property (P6). Then R is commutative.*

**Proof of theorem 2.6**: By the hypothesis of (P6), we have

F(x ◦ y) = F(x) ◦ y − d(y) ◦ x for all x, y I. (19)

Replacing y by yx in (19) and using (P6), we find that

(x ◦ y) d(x) = − y [F(x), x] − y(d(x) ◦ x) + [y, x] d(x) for all x, y I. (20)

For any r R, replacing y by ry in (20) and applying (20), we get

2[x, r] y d(x) = 0 for all x, y I.

Since R is 2-torsion free, we obtain [x, r] y d(x) = 0 for all x, y I and r R.

Therefore, [x, R] I d(x) = 0 for all x I.

Hence, by virtue of Lemma 2.2 we get R is commutative.

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*M. Rais Khan, Deepa Arora & M. Ali Khan  
Department of Mathematics,   
Jamia Millia Islamia (Central University),  
New Delhi-110025, India.*[*mohdrais\_khan@yahoo.co.in*](mailto:mohdrais_khan@yahoo.co.in)*,* [*musk.deepa@gmail.com*](mailto:musk.deepa@gmail.com)*,* [*mkhan91@gmail.com*](mailto:mkhan91@gmail.com)