**Solving Two Point Boundary Value Problems for Ordinary Differential Equations by a New Approach: Exponential Finite Difference Method**

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**Abstract:** In this article , a new exponential finite difference scheme for the numerical solution of two point boundary value problems with Dirichlet's boundary conditions is proposed. The scheme is based on an exponential approximation of the Taylor expansion for the discretized derivative .The convergence of the scheme discussed under appropriate condition .The theoretical and numerical results show that this new scheme is efficient and at least fourth order accurate.

**Key words** : Convergent exponential method , Finite difference method , Fourth order method ,Mac Lauren series expansion, Boundary value problems ,Maximum Absolute error.

**AMS 2000 Sub. Classification:** 65L05 , 65L12

1**. Introduction :**

 In this article ,we implement an exponential finite difference method for solving the two point non-linear boundary value problem of the form

subject to boundary conditions

The existence and uniqueness of the solution to problem (1.1) is assumed. Further we assume that problem(1.1) is well posed with continuous derivatives and that the solution depends differentially on the boundary conditions. The specific assumption on to ensure existence and uniqueness will not be considered [1,2,3].

In this article , we develop a new algorithm capable of solving equation of form (1.1). To best of our knowledge, no similar method for the solution of problem (1.1) has been discussed in literature so for. In this paper we discuss exponential finite difference, a new method of at least order four based on local assumption. Its development and analysis is based on Taylor, Mac Lauren and Exponential series expansion.

 In the next section we discuss the derivation of the exponential finite difference method. A local truncation error and convergence of the method discussed in Section 3 and finally the application of the developed method to the problem (1.1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 4. A discussion and conclusion on the performance of the method are presented in section 5.

2. **Derivation of the method :**

 We define ,the finite numbers of nodal points of the interval , in which the solution of the problem (1.1) is desired as

using uniform step length where , and b .Suppose we have to determine a number ,which is numerical approximation to the value of the theoretical solution of the problem (1.1) at the nodal point and other similar notations like defined as Assuming the local assumption that no previous truncation errors have been made in computation of solution at mesh point , i.e. . Following the ideas in [4,5] , we propose an approximation to the theoretical solution of the problem (1.1) exponential finite difference scheme as ,

where are unknown constants and ,an unknown sufficiently differentiable function of step length h.

Let us define a function and associate it with (2.4) as,

Let can be expand in Mac Lauren's series. So we write in Mac Lauren series we have,

The application of ( 2.6) in expansion of will provide an approximation of the form as

Expand in Taylor series about mesh point and using (2.7) in it, we have

where etc similar to notation in [6], and comparing the coefficients of in (2.8) ,we will get following system of nonlinear equations

To determine unknown constants in (2.9) ,we have to assign arbitrary values to some constants . So to simplify above system of equations, we have considered following

Uusing (2.10) in (2.9) and solving the reduced system of equations ,we obtained

Substituting the values of etc. from (2.10) and (2.11) in (2.6), we have

Finally substitute the values of , ,and from (2.11) and (2.12) in (2.4),we will obtain our proposed exponential finite difference method as

If we write system equations given by (2.13) for each nodal point , we will obtain the nonlinear system of equations and linear system of equations in case when source function is .

For computational purpose reported in Section 4 ,we have used second order finite difference approximation in place of

**3. Local Truncation Error and Convergence :**

The truncation error at the nodal point may be written as in [7,8,9],

The exact solution of (1.1) satisfies the equation

Subtracting (2.15) from (2.13) and applying the mean value theorem,

Let us substitute in above expression and simplify ,we have

Let us write (3.16) in matrix form

where , ,

 and

Also we have for , so and

So we have

It follows from (3.17) that as Thus we conclude that method (2.13) converges and the order of convergence is at least four.

4. **Numerical Results:**

 To illustrate our method and demonstrate computationally its efficiency ,we consider some model problems. In each case we took uniform step size h.. In tables ,we have shown maximum absolute error MAU and l2-norm of the error ERR , computed for different values of N and these are defined as ,

and

 We have used Newton-Raphson iterationmethod to solve the system of nonlinear equations and iterative method to solve linear system of equations.

 All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC .The solutions are computed on N-1 nodes and iteration is continued until either maximum difference between two successive iterates is less than ornumber of iteration reached .So we have given in table the no. of iterations performed to achieve desired accuracy.

**Example1.** The first model problem is nonlinear and consists in solving the equation

in interval [0,1] ,with the boundary conditions for which the analytical solution is found to be .The MAU and ERR for different values of N are presented in Table 1.

**Example2.** The second model problem is linear and consists in solving the equation

in interval [0,1] ,with the boundary conditions for which the analytical solution is found to be .The MAU and ERR for different values of N and are presented in Table 2.

**Example3.** The third model problem is highly sensitive nonlinear, well known Troesch's equation [10] and consists in solving the equation

in interval [0,1] ,with the boundary conditions for which the analytical solution is considered to be .The MAU for different values of N and are presented in Table 3.

**Example4.** The fourth and final model problem is nonlinear and consists in solving the equation

in interval [0,1] ,with the boundary conditions for which the analytical solution is considered to be .The MAU for different values of N are presented in Table4.

Table1.

|  |  |
| --- | --- |
| Method (2.13) | N |
| Iterations | ERR | MAU |
| 12 | .39054255(-2) | .35644150(-2) | 4 |
| 27 | .26143758(-3) | .23323059(-3) | 8 |
| 38 | .11079033(-4) | .13647080(-4) | 16 |
| 02 | .13229610(-7) | .12704658(-6) | 32 |

Table2.

|  |  |
| --- | --- |
| Method (2.13) | N |
| Iterations | ERR | MAU |
| 123 | .10031415(0) | .27641535(-1) | 4 |
| 259 | .44306965(-2) | .12654066(-2) | 8 |
| 296 | .12922547(-3) | .60677528(-4) | 16 |
| 03 | .88322334(-8) | .59604645(-7) | 32 |

Table3.

|  |  |
| --- | --- |
| Method (2.13) |  N |
|  |  |  |
| Iterations | MAU | Iterations | MAU | Iterations | MAU |
|  | .. |  | .. | 4 | .33518672(-3) | 8 |
| 1 | .52547678(-2) | 3 | .16759336(-3) | 7 | .34719706(-4) | 16 |
| 4 | .78815967(-4) | 6 | .17359893(-4) | 15 | .10840595(-5) | 32 |
| 8 | .27474016(-5) | 14 | .54202974(-6) | 12 | .11920929(-6) | 64 |
| 21 | .83819032(-7) | 12 | .59604645(-7) | 5 | .29802322(-7) | 128 |

Table4.

|  |  |
| --- | --- |
| Method (2.13) | N |
| Iterations | ERR | MAU |
| 20 | .50859000(-1) | .39065361(-1) | 4 |
| 54 | .37111132(-2) | .25463104(-2) | 8 |
| 129 | .24020411(-3) | .16093254(-3) | 16 |
| 79 | .34110020(-5) | .53644180(-5) | 32 |
| 17 | .21216829(-7) | .11992816(-6) | 64 |

5. **Conclusion :**

 A new approach to obtain numerical method for the numerical solution of second order boundary value problems has been developed. The new scheme has advantages and disadvantages when considered individually. For example ,the scheme based on exponential approximation , if source function *f (x)* then system of equation is linear otherwise nonlinear system of equations ,which is always difficult to solve. On the other hand method has good rate of convergence and less discretization error. The method is same as Numerov Method if we consider second order approximation to exponential function. It is an advantage of the method that we consider exponential function without any approximation in computation. This mean method depend on machine epsilon and software used in computation. It may be noted that method can be avoided in the case where the source function vanishes in the computational domain of the problem.

The decision to use a certain difference scheme does not only depend on the given order of the method but also on its computational efficiency. The numerical results for model problems show that method is computationally efficient. Also it is observed from the results that method has higher accuracy i.e. smaller discretization error.

In the present article different form of high order method has been derived on the basis of exponential function and local assumption, which was applied to approximate the second order boundary value problems. It is not clear how this local assumption affect the overall solution of the problem. Investigation in this direction will be done in the future. In addition the development of this exponential method ,will lead to possibility to approximate higher order derivatives in term of power of its lower order derivatives, to raise the order and accuracy of the method. Work in this direction is in progress.

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