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TWO-COLUMN PROOFS – A METHOD OF PROVING THEOREMS IN GEOMETRY: Appropriations in the Brazilian context

Provas em duas colunas – um método de demonstrar teoremas em geometria: apropriações no contexto brasileiro

Pruebas en dos columnas – un método para demostrar teoremas en geometría: apropiaciones en el contexto brasileño

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Abstract: This article investigates the creation, circulation, and appropriation of the demonstration method known as the *two-column proof*, widely used in the teaching of geometry in the United States since the early 20th century. Drawing on Herbst (2002), we discuss how this method became consolidated in the U.S., amidst a transition between different forms of presenting proofs in school textbooks. We identify textbooks produced during the Modern Mathematics Movement (MMM) as possible initial vectors for introducing the method in Brazil. We argue that the incorporation of the two-column proof reflects a process of restructuring demonstrative practices within school mathematics. The study contributes to the history of school geometry by showing that, throughout the 20th century, mathematical proof ceased to be understood solely as a practice of specialists and came to be viewed as a skill to be learned in basic education.
Keywords: textbooks; Modern Mathematics Movement; History of school geometry.

Resumo: Este artigo investiga a criação, circulação e apropriação do método de demonstração conhecido como prova em duas colunas, amplamente utilizado no ensino de Geometria nos Estados Unidos desde o início do século XX. A partir do estudo de Herbst (2002), discutimos como o método se consolidou naquele país, em meio à transição entre diferentes formas de apresentação das demonstrações nos livros didáticos. Identificamos os livros produzidos durante o Movimento da Matemática Moderna (MMM) como possíveis vetores iniciais de introdução do método no Brasil. Argumentamos que a incorporação da prova em duas colunas indica um movimento de reestruturação das práticas demonstrativas no âmbito escolar. O estudo contribui para a história da Geometria escolar ao evidenciar que a demonstração, ao longo do século XX, deixou de ser entendida apenas como prática de especialistas e se constituiu como habilidade a ser aprendida no ensino básico.
Palavras-chave: livros didáticos; Movimento da Matemática Moderna; História da geometria escolar.

Resumen: Este artículo investiga la creación, circulación y apropiación del método de demostración conocido como *prueba en dos columnas*, ampliamente utilizado en la enseñanza de la geometría en los Estados Unidos desde principios del siglo XX. A partir del estudio de Herbst (2002), discutimos cómo dicho método se consolidó en ese país, en medio de la transición entre diferentes formas de presentación de las demostraciones en los libros de texto. Identificamos los libros producidos durante el Movimiento de la Matemática Moderna (MMM) como posibles vectores iniciales de introducción del método en Brasil. Argumentamos que la incorporación de la prueba en dos columnas indica un proceso de reestructuración de las prácticas demostrativas en el ámbito escolar. El estudio contribuye a la historia de la geometría escolar al mostrar que, a lo largo del siglo XX, la demostración dejó de entenderse únicamente como una práctica de especialistas y pasó a constituirse como una habilidad a ser aprendida en la educación básica.
Palabras clave: libros de texto; Movimiento de la Matemática Moderna; Historia de la geometría escolar.

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Introduction

This study aims to historically analyze a method of proving theorems in geometry known as two-column proofs[1]. We begin by presenting a summary of Herbst's (2002) study, which discusses the creation and establishment of this method in the United States of America (USA) since the beginning of the 20th century. Next, we examine a set of Brazilian textbooks published between 1930 and 1950 to verify its presence during this period. Finally, we analyze what we consider to be the first insertions of the method in Brazilian textbooks, within the context of the Modern Mathematics Movement (MMM)[2].

It is important to note that the research path did not develop linearly. At the end of the 20th century, a group of professors from PUC/SP, working in Geometry courses in the initial training of Mathematics teachers, used the two-column proof method with undergraduate students, without, however, knowing its origin, its arrival in Brazil, or the historical context that favored its appropriation. At that time, these teachers were not familiar with studies in the History of Mathematics Education or with the MMM itself.

Years later, in 2006, within the scope of a Brazil-Portugal International Cooperation project[3] focused on the study of Modern Mathematics, the approach to Geometry in the first collection of modern textbooks, by Osvaldo Sangiorgi[4], was the subject of investigation. However, in that study, the two-column proof method did not receive specific attention, nor were textbooks prior to the 1960s examined.

The return to this topic occurred in 2023, within the framework of a new research project[5], that now focuses on the practices of proof and validation in Geometry. The audience to whom these proposals were originally directed consisted of students approximately 13 years old, particularly those using textbooks for the 3rd

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[1] Throughout the text, the method is explained and exemplified.

[2] The MMM (Modern Mathematics Movement) was an international movement, initiated in the 1950s, whose objective was to discuss and develop proposals for change in mathematics education, both in Europe and America. The proposals advocated for the unification of the various fields of mathematics, bringing basic education closer to higher education, which corresponds to the language and structure used by mathematicians of the time. In Brazil, it gained momentum from the 1960s onwards.

[3] CAPES/GRICES Project (2006-2010) *Modern mathematics in schools in Brazil and Portugal: comparative historical studies*.

[4] Osvaldo Sangiorgi (1921-2017), a mathematician trained at the University of São Paulo (USP), was a renowned mathematics professor and author of textbooks since the 1950s. He is considered a leading figure in the introduction and dissemination of modernizing ideas for mathematics education in Brazil. In 1961, he created the Mathematics Education Study Group (GEEM), responsible for the first initiatives in the circulation of Modern Mathematics in Brazil, and was also the first textbook author to incorporate the modern approach into textbooks for middle school courses (students aged 11 to 14) in the 1960s.

[5] Research Project History of Geometry in Education and the Modern Mathematics Movement, funded by Fapesp (2023/04639-8) under the coordination of the article's author.

year of secondary school following the Francisco Campos Reform (1931). In 2024, during the 26th ICMI *Advances in Geometry Education*, we came into contact with the study by Patrício Herbst (2002), who indicated that the two-column proof method constituted a long-standing standard in North American schools, while in Brazil it began to appear as a novelty from the 1960s onwards.

These aspects reveal that the methodological paths of a research project involve thematic choices, availability and nature of sources, as well as dialogues with appropriate theoretical frameworks. We consider it relevant to highlight that this trajectory played a fundamental role in the formulation of the project *Validation Process in Geometry Contexts: Histories and Articulations with Teacher Training*[6]. We use the term *validation process* instead of "proofs" or "demonstrations" to emphasize that, throughout schooling and in different historical periods, multiple ways of justifying, explaining, arguing, or supporting geometric properties and concepts can be identified.

It is important to consider that the topic of argumentation, closely related to methods of proving theorems, has been the subject of study in various fields of knowledge. These studies have fostered connections and enabled the construction of cooperative networks among researchers in Brazil and abroad[7], which reinforces the need to bring to the debate a historical trajectory regarding the processes of creation, circulation, and appropriation of different ways of arguing – as is the particular case of the two-column proof method.

Thus, the present study constitutes a historical narrative about the incorporation of the two-column test method, delimiting the period of analysis between the beginning of the 20th century – with the creation of the method in the USA – and its appropriation in Brazilian textbooks during the MMM (Modern Mathematical Movement), in the 1960s and 1970s.

From a theoretical and methodological point of view, we adopted the framework of the circulation and appropriation of knowledge, considering textbook authors and teachers as cultural mediators in the production and dissemination of pedagogical practices (Matasci, 2016). The cultural transfer approach allows us to understand that appropriation processes depend on the contexts of reception, highlighting that the knowledge that arrives is not identical to the knowledge emitted (Burke, 2016).

The main Brazilian sources analyzed to examine the insertion of a new method of proof in textbooks – the two-column proof method – were the regulations from different periods, which generally expressed the proposal of a deductive study, without, however, specifying or presenting examples. In addition to the analysis of these regulations, a comparison of textbooks was carried out, allowing for the

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7 The Brazilian Argumentation Association (ABA) was created in 2023. More details can be found on the website: https://www.ababrasil.net/

interpretation of educational proposals that may reveal aspects of school culture, as emphasized by Munakata (2012):

> The notion of school culture refers not only to norms and rules—whether explicit or implicit—symbols and representations, and prescribed knowledge, but also, and above all, to practices, processes of appropriation, the attribution of new meanings, and forms of resistance that produce multiple and varied configurations within the school context. [...] One of the elements particularly characteristic of the school is precisely the textbook (p. 122).

In this sense, a critical analysis of textbooks from different periods allows us to identify problems and get closer to pedagogical practices, as well as the multiple attempts to attribute meaning to the processes of argumentation, validation, and justification—aspects that can be considered part of the educational objectives present in different areas, levels, and school contexts.

Two-column proof method in the USA

Patrício Herbst (2002), in his article Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century[8], presents a historical account that allows us to understand how the two-column proof method contributed to strengthening the teaching of Geometry in American schools since the end of the 19th century.

The author identifies three distinct periods in the teaching of geometric proofs or demonstrations[9] in secondary school, based on the analysis of textbooks, which he names: (1) Era of the Text: Proofs were presented in their entirety in books. Students studied and reproduced complete demonstrations; (2) Era of the Original: Books began to include problems at the end of chapters so that students could produce "original" proofs. The expectation was that students would think and reason, even though the proof was not treated as a systematically taught skill; and (3) Era of the Exercise: Proofs began to be organized into two columns (statements and justifications). The demonstration became an explicit teaching object, a skill to be learned and practiced.

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8 Establishing a standard for proofs in American school geometry: evolution of two-column proofs in the early 20th century.

9 Herbst (2002) uses the term proof as a synonym for demonstration on several occasions. The same usage is observed in references from the Committee of Ten—a group of educational leaders chaired by Charles Eliot during the 19th century. This terminological equivalence also appears in the classifications proposed by the author, such as the Era of the Text, the Era of the Original, and the Era of the Exercise.

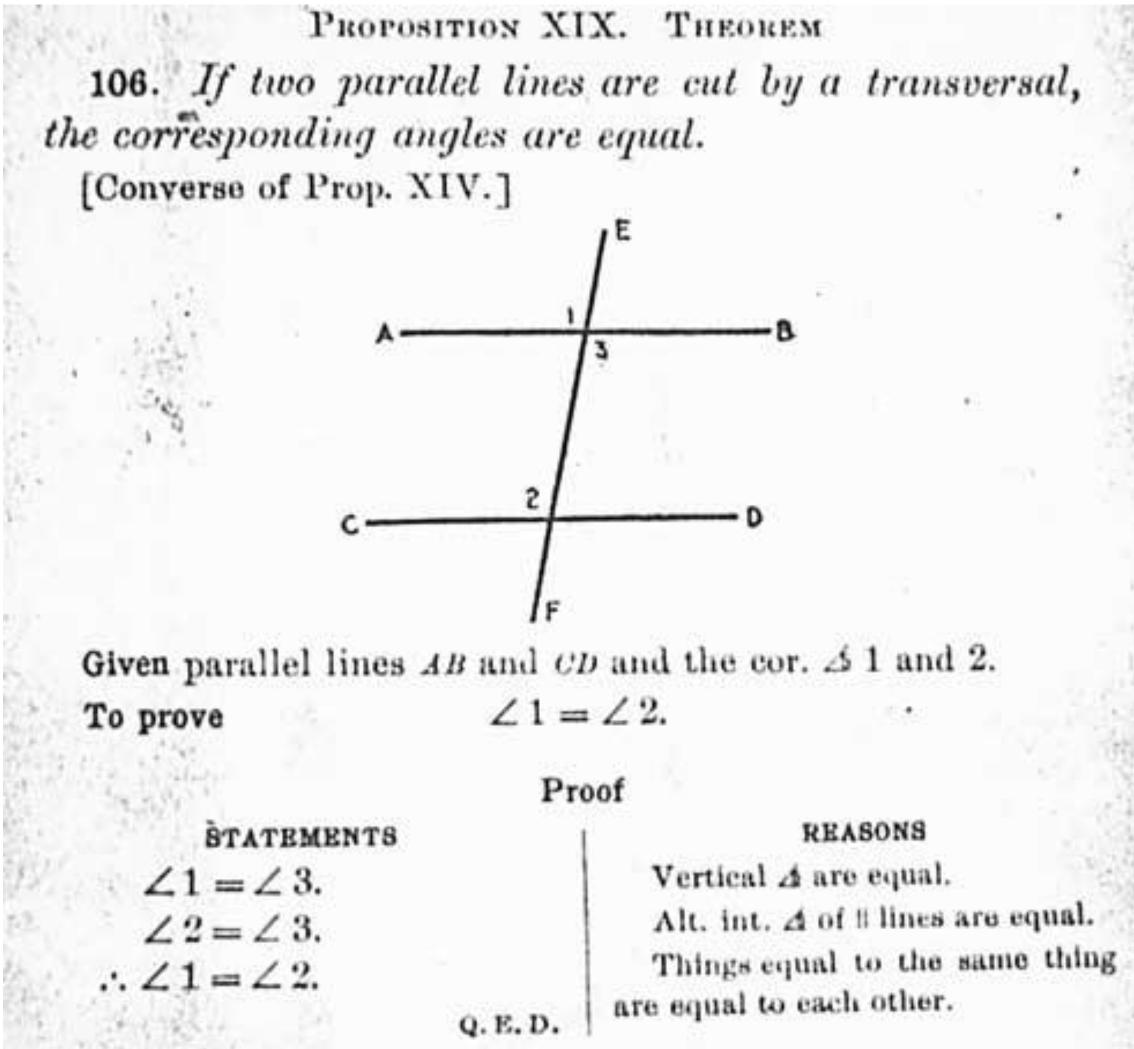
In the Era of the Text, the geometric knowledge and the ability to prove theorems were indistinguishable skills; proofs were written in long paragraphs and often in complex language. However, as the teaching of geometry expanded in secondary schools, the approach began to move away from the simple reproduction of a geometry text (Herbst, 2002). Everything indicates that, linked to this conception, was the proof of a theorem as something performed by mathematicians (geometers) and possibly in only one way, according to the sequence presented in textbooks, with the student's role being to memorize and reproduce the demonstration.

In the Era of the Original, authors began including exercises at the end of chapters as opportunities for students to produce original work. Authors created representations for conducting proofs, such as drawing diagrams. Wentworth (1878) is credited with pioneering the visual reorganization of the textbook page: he presented each theorem and its proof on a single page, introducing short justifications between the steps of the argument. This arrangement made the steps of logical reasoning explicit, allowing the teacher to verify if the student had understood the argument. The justification for each step was indicated in small print between that step and the next, avoiding the need to interrupt the argument process by referring to a previous section. Furthermore, each distinct statement in the proofs began on a new line. We can infer that this was an important moment in seeking a pedagogical way to present the steps: each sentence on a line, the entire proof on the same page, and inserting justifications explaining the steps, even if in small print.

The transition to the Era of Exercise marked the systematization of the teaching of proofs. Textbooks began to explicitly define what was considered a proof: each statement should be based on a postulate, an axiom, a definition, or some previously demonstrated proposition. Authors such as Wooster Beman and David Eugene Smith, in 1895, developed introductory sections on the nature of geometric proof, presenting examples and guiding the student through gradual steps.

The geometry textbooks by Arthur Schulze and Frank Sevenoak (1913) were the first to explicitly present proofs in two columns, with statements and justifications separated by a vertical line, as illustrated in Figure 1:

Figure 1 - Two-column proof by Schulze and Sevenoak (1913)



PROPOSITION XIX. THEOREM

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]

Given parallel lines *AB* and *CD* and the cor. ∡ 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
$\angle 1 = \angle 3$.	Vertical ∡ are equal.
$\angle 2 = \angle 3$.	Alt. int. ∡ of ∥ lines are equal.
$\therefore \angle 1 = \angle 2$.	Things equal to the same thing are equal to each other.

Q. E. D.

Note. From *Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century* (Herbst, 2002, p. 284).

The main difference between the Era of the Original and the Era of the Exercise lay in the volume and nature of the tasks: while the former involved a small number of difficult problems, the latter proposed that students solve numerous simple proofs that were carefully graded in difficulty. The authors then faced the challenge of balancing fundamentals and exercises: which propositions should be considered basic? In what order should they be presented? Which proofs should be models and which should be performed by the students?

In this context, the two-column proof model has become established as a pedagogical resource, allowing the logical structure of argumentation to be highlighted and the links between statements and justifications to be made visible. According to Herbst (2002), the method represented an achievement for the teaching of Geometry in the 20th century, remaining as an icon of American school practice for several decades.

DEDUCTIVE GEOMETRY IN BRAZIL

In the 19th century and the first decades of the 20th century, the Brazilian secondary education had the *Colégio Pedro II*[10] in Rio de Janeiro as an important formative reference. Studies such as Valente (2007) and Bittencourt (1993) show that the textbooks adopted by this institution predominantly followed French models, often in the form of translations or compilations. A significant example is the book by Cristiano Benedito Ottoni, recommended for the subject of Geometry and considered a reference in the mid-19th century, which is based on the French work of Vincent (Valente, 2007). The second edition, published in 1857, presents approximately 280 definitions and theorems, demonstrated by three distinct methods: direct proof, proof by contradiction, and superposition. However, there are no exercises for students or proposals for producing proofs. Thus, this material can be associated with what Herbst (2002) characterizes as the Era of the Text, in which proofs are fully provided, and the student performs an essentially reproductive function.

With the Francisco Campos Reform in 1931, which structured secondary education in Brazil, the intuitive method began to be advocated as the basis of the first movement to modernize mathematics teaching, led by Euclides Roxo[11]. In the case of Geometry, the 1931 Program instituted, for the first two years of secondary school, an experimental and intuitive approach (Geometric Initiation), from which one should evolve to formal deductive exposition. Roxo justified this proposal as follows:

> From a psychological point of view, the preparatory course in geometry is justified not only as a bridge between ordinary spatial experience and deductive geometry, but also as a complement to the latter, which, while it can develop the capacity for deductive reasoning, it is incapable, by itself, of completing the student's mathematical education. Indeed, what is called intuitiveness or spatial perception of the environment, an aptitude eminently necessary for success in practical life, cannot be provided by demonstrative geometry alone. On the contrary, the exclusive preoccupation with logical deduction prevents what could be called the spatial acclimatization of the individual (Roxo, 2003, p. 186).

However, this modernizing proposal faced strong resistance, including from colleagues at *Colégio Pedro II* itself. Professor Almeida Lisboa, for example, publicly

[10] The Imperial *Colégio Pedro II* was created in 1837 with the intention of serving as a model for secondary education in Brazil (Valente, 2007). In 1889, with the Proclamation of the Republic, the name was changed to National Institute of Secondary Instruction, and later to National Gymnasium. In 1911, it reverted to the name *Colégio Pedro II*.

[11] Euclides Roxo, engineer, professor, intellectual, and mathematics educator in the early 20th century. More details can be found in Dassie & Carvalho, 2010.

criticized Roxo's books in the *Jornal do Commercio* (1931), Rio de Janeiro, arguing that the excessive emphasis on intuitive activities had suppressed the deductive character considered essential to the study of Mathematics:

> The books in which Mr. Euclides Roxo presents his program are excessively childish. Their practical applications are illusory and of no scope. In them there is no trace of the simplest demonstration of any theorem, however elementary: there are only material verifications, and therefore imperfect and crude ones. The exemplary reasoning, characteristic of a demonstration and of Mathematics itself, has disappeared (Almeida Lisboa *apud* Carvalho, 2003, p. 131).

Thus, although the 1931 Reform advocated for intuitive geometry in the early years of secondary school, criticism and resistance contributed to the deductive approach remaining dominant in teacher training throughout the first half of the 20th century. This characteristic becomes even more evident in the 1950s, when the National Congresses on Mathematics Education (CNEM) began[12]. In the Proceedings of the II CNEM (1957), professors such as Antonio Rodrigues[13] and Benedito Castrucci[14]pointed to the excessive memorization of proofs as one of the main problems in teaching practice, suggesting a reduction in the number of theorems proven and the inclusion of forms of experimental or intuitive proof.

A study conducted by Oliveira and Pietropaolo (2008, p. 105) on articles about Mathematics and Drawing published in the *Revista Escola Secundária* between 1957 and 1963 indicated that:

> In summary, these authors agreed with the prescribed programs that deductive geometry should be developed in classes, respecting the pace, needs, and interests of the students, provided that the teachers' practices did not stray too far from the maxim: the proofs must be rigorous, given that a proof is crucial in the mathematical culture.

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[12] The National Congresses on Mathematics Education took place between the 1950s and 1960s. The first was held in Salvador, Bahia, in 1955; the second in Porto Alegre, Rio Grande do Sul, in 1957; the third in Rio de Janeiro, Rio de Janeiro, in 1959; the fourth in Belém, Pará, in 1962; and the fifth and last in São José dos Campos, SP, in 1966.

[13] Professor of Geometry at the Faculty of Philosophy of UFRGS. His thesis is presented in the Annals in the chapter entitled "The Teaching of Deductive Geometry".

[14] Professor at the Faculty of Philosophy, University of São Paulo. PhD in Mathematics. Author of geometry textbooks and responsible for Geometry at GEEM.

In summary, despite attempts at renewal, everything indicates that the deductive geometry remained as the structural basis of teaching in Brazilian secondary education at least until the mid-20th century, being formally introduced in the 3rd year of middle school, intended for students of approximately 13 years of age.

Tests in textbooks from 1930 to 1950 in Brazil

A previous study (Jahn & Leme da Silva, 2023) identified, in educational regulations between 1930 and 1950, that deductive geometry was introduced from the 3rd year of secondary school onwards in the three reforms of the period: Francisco Campos (1931), Capanema (1942), and Simões Filho (1951). In the first two, the intuitive approach constituted a preparatory stage; in the last, teaching became exclusively deductive.

For this study, we selected representative textbooks from each decade, prioritizing widely circulated works with multiple editions and authored by recognized professors. The authors analyzed—such as Euclides Roxo, Cecil Thiré, Júlio César de Mello e Souza (Malba Tahan), Algacyr Maeder, Ary Quintella, Carlos Galante, Oswaldo Santos, and Osvaldo Sangiorgi—worked at central institutions, such as *Colégio Pedro II* and *Colégio Militar*, and also received their education at the Faculty of Philosophy of the University of São Paulo.

The intention is not to generalize the results, but to examine whether the two-column proof method appears in textbooks and what opportunities exist for conducting proofs[15] were offered to students. To achieve this, we counted the approximate number and analyzed the exercises that asked students to demonstrate or prove theorems, from the textbooks of the 3rd year of middle school, the year in which deductive geometry began, as shown in Table 1. The questions found generally ask for exercises such as: “Demonstrate that the angle bisectors of the base angles of an isosceles triangle are equal” (Roxo et al, 1944, p. 180) or “Prove that the angle bisectors of angles that have parallel sides are parallel or perpendicular.” (Galante et al, 1953, p. 236).

[15] In all six books analyzed, we found the terms "proof" and "demonstration" used synonymously. However, the expression "to demonstrate" or "demonstration of the theorem" appears much more frequently, especially in the 1950s.

Table 1 – Mathematics textbooks

Reform	Year	Book	Authors	Publisher	Examples of evidence	Two-column proof
Francisco Campos	1934	Mathematics Course 3rd Year	Euclides Roxo, Cecil Thiré, Mello e Souza	Francisco Alves Bookstore	6	0
	1936	Math Lessons 3rd Year	Algacyr Munhoz Maeder	Companhia Melhoramentos de São Paulo	0	0
Capanema	1944	Middle School Mathematics 3rd Series	Euclides Roxo, Cecil Thiré, Mello e Souza	Livraria Francisco Alves	70	0
	1946	Mathematics 3rd Year	Ary Quintella	National Publishing Company	46	0
Simões Filho	1953	Mathematics Third Series	Carlos Galante, Osvaldo Marcondes dos Santos	Editora do Brasil S/A	19	0
	1958	Mathematics Course 3rd Year	Osvaldo Sangiorgi	National Publishing Company	84	0

Note. From *Curso de matemática: 3º ano* (Roxo et al., 1934); *Lições de matemática (3º anno [3ª série])* (Maeder, 1936); *Matemática ginasial: 3ª série* (Roxo et al., 1944); *Matemática: 3º ano* (Quintella, 1946); *Matemática: Terceira série* (Galante & Santos, 1953); and *Matemática para a terceira série ginasial* (Sangiorgi, 1958).

The first result of our investigation was that none of the six textbooks examined presented the two-column proof model, in contrast to Herbst's (2002) report, which indicates the presence of this model as a way of systematizing proofs in school textbooks in the United States since the beginning of the 20th century. For some reason, the model was not incorporated into Brazilian textbooks during the period analyzed, at least not among those selected for this study.

However, when examining the practice of assigning proof and demonstration exercises to 3rd-grade secondary school students (approximately 13 years old), we identified changes throughout the period considered. In the 1930s, during the Francisco Campos Reform, which introduced the proposal of intuitive geometry in the first two grades, the presence of exercises is quite reduced, both for general content and for the performance of theorem proofs.

From the 1940s onwards, with the Capanema Reform, a distinct scenario is observed: there is a significantly greater number of exercises in the books analyzed.

In the two books by Roxo et al., one from 1934 and the other from 1944, despite sharing the same authors and differing in publication date by ten years, we find a significant increase in the number of exercises that require proof, rising from 6 in the 1930s to 70 in the 1940s. Roxo et al. (1944) used the expressions "show," "prove," and "demonstrate" as synonyms, and the exercises were presented both in general lists at the end of the chapters and in specific sections entitled "theoretical exercises."

In the 1950s, with the Simões Filho Reform, we see the maintenance of this emphasis on demonstration. In Sangiorgi's book (1958), which also uses the expressions "prove" and "demonstrate" as equivalents, the number of exercises intended for proof is even greater than in previous decades.

In summary, we observed a growing emphasis on exercises demonstrating theorems and properties in textbooks from the 1950s, which aligns with the debates recorded at the II National Congress on Mathematics Education (1957), where the difficulties attributed to the teaching of Geometry were highlighted, especially the fact that students memorized the proofs. The progressive increase in the number of proof exercises between the 1930s and 1950s can be interpreted as a transition from what Herbst calls the Era of the Original to the Era of the Exercise.

Two-column test method in Brazilian textbooks

At the IV National Congress on Mathematics Education (CNEM), held in Pará in 1962, the Mathematics Education Study Group (GEEM) presented a mathematics program conceived from a modernizing perspective. The document, entitled *Minimum Topics for a Modern Mathematics Program*, was unanimously approved by the congress participants. For the 3rd year of middle school (currently the 8th grade of elementary school, students around 13 years old), the topic *Fundamental Elements of Plane Geometry* was accompanied by methodological guidelines, such as: “intuitively introduce the fundamental elements and their properties; always use the language of sets and their operations; and show how some properties are consequences of other more elementary ones, introducing the deductive process” (GEEM, 1965, p. 94). Thus, it is observed that the introduction of deductive geometry, even in the context of Modern Mathematics, was not questioned; on the contrary, it was reaffirmed, even if preceded by an intuitive approach.

With the enactment of the Law of Guidelines and Bases of National Education (LDB No. 4,024/61), the State Councils of Education gained autonomy to establish curricular programs. In this context, in 1965, the state of São Paulo published the document *Suggestions for a Curricular Program Outline for the Mathematics subject*, in which deductive geometry remained concentrated in the 3rd and 4th years of secondary school (students aged 13 and 14).

Throughout the 1960s, the Modern Mathematics Movement (MMM) consolidated itself through the activities of the CNEM (National Center for Modern Mathematics), the teacher training courses offered by the GEEM (Mathematics Education Study Group), and other initiatives that contributed to the dissemination of its proposals. In this context, the first textbooks aligned with the modern approach began to emerge.

To analyze this period, we selected two works considered representative of the first insertions of Modern Mathematics in the Brazilian context (Table 2). One of them, authored by Osvaldo Sangiorgi, coordinator of GEEM, is responsible for the 3rd year of middle school (13-year-old students). The second book, dedicated exclusively to Geometry, is a translation of a North American work, with a complete study of plane and spatial Geometry, covering the geometry curriculum from the 3rd year of middle school to high school.

Table 2 – Mathematics and Geometry textbooks

Program	Year	Book	Authors	Publisher	Two-column proof
Suggestions for a script of Program SEE-SP (1965)	1967	Modern Course in Mathematics 3rd Year	Osvaldo Sangiorgi	National Publishing Company	5
	1971	Modern Geometry – Part I	Moise & Downs Translated by Renata Watanabe and Dorival Mello	Edgard Blücher Ltda	30

Note. From *Matemática: Curso moderno (3º volume para os ginásios)* (Sangiorgi, 1967); and from *Geometria moderna – Parte I* (Moise & Downs, 1971).

Osvaldo Sangiorgi was a widely recognized author in the textbook market, with works published by *Companhia Editora Nacional* since the 1950s (Valente, 2008). One of the books included in Table 1 presents 84 exercises of "proving" or "demonstrating," terms used synonymously by the author. The book *Matemática–Curso Moderno*, intended for the 3rd year of secondary school (1967), was analyzed comparatively with the same author's book for the same grade, published in 1958, in the article *Não decore provas de teoremas! A Geometria Moderna de Osvaldo Sangiorgi* (Jahn & Leme da Silva, 2023). The authors concluded that, in the modern book:

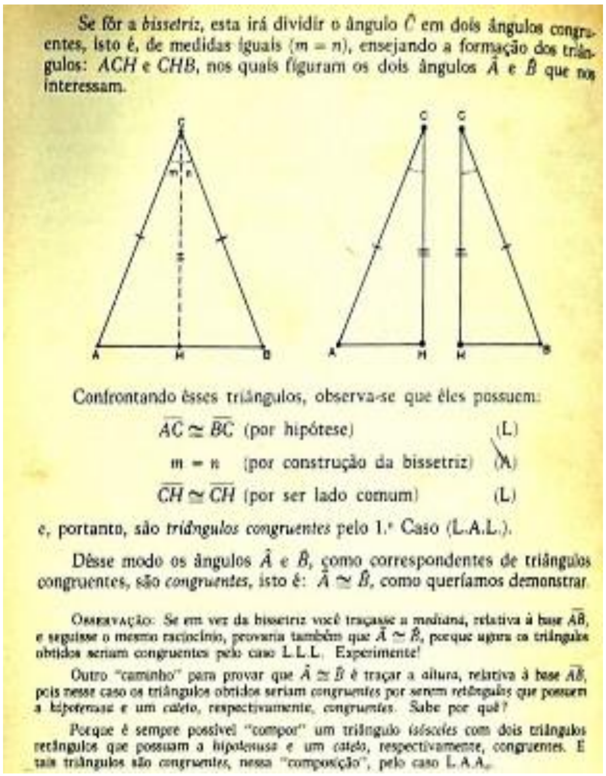
> This explains the need to reduce the number of theorems demonstrated in the modern collection, since the treatment of proofs gains a different, much more explanatory perspective, mobilizing different registers of representation for logical chaining. We can infer that these innovations correspond to the way Sangiorgi created and elaborated, from the moment he was immersed in the MMM debate, so that a 13-year-old can understand

the deductive process instead of memorizing the theorems and their proofs (Jahn & Leme da Silva, 2023, p. 17).

Among the innovations, the introduction of a preparatory stage of intuitive geometry before deductive study stands out, with exploratory exercises in which the term "prove" is used in quotation marks. In the modern collection, unlike the previous collection, proving is not synonymous with demonstrating: "proving" refers to an intuitive verification—for example, identifying, through geometric constructions, the cases of triangle congruence—while demonstrating is reserved for formal deductive study. Furthermore, the number of exercises requiring proofs is significantly reduced: from 84 in the 1958 book to 29 in the 1967 book.

Another innovative feature is the presentation of multiple representations of proofs for the same theorem. Sangiorgi presents five representations of the logical chain for Theorem 1 of the book— on the congruence of the base angles of an isosceles triangle. The first three correspond to usual proofs, each using a different auxiliary construction: the bisector of the angle opposite the base, the median, and finally, the altitude relative to the base, configuring three distinct ways of proving the same theorem (Figure 2). The fourth representation uses the two-column proof model, and the fifth employs the model of “drawn diagrams,” colored and organized with a series of deductions through constructions (–), equivalences (⇔), and implications (⇒), a technique preferred by the French mathematician and pedagogue Lucienne Félix (Figure 3).

Figure 2 – Three usual ways to prove Theorem 1



Se fôr a *bissetriz*, esta irá dividir o ângulo $\hat{C}$ em dois ângulos congruentes, isto é, de medidas iguais ($m = n$), ensejando a formação dos triângulos: *ACH* e *CHB*, nos quais figuram os dois ângulos $\hat{A}$ e $\hat{B}$ que nos interessam.

Confrontando êsses triângulos, observa-se que êles possuem:

$\overline{AC} \simeq \overline{BC}$ (por hipótese) (L)

$m = n$ (por construção da bissetriz) (A)

$\overline{CH} \simeq \overline{CH}$ (por ser lado comum) (L)

e, portanto, são *triângulos congruentes* pelo 1.º Caso (L.A.L.).

Dêsse modo os ângulos $\hat{A}$ e $\hat{B}$, como correspondentes de triângulos congruentes, são *congruentes*, isto é: $\hat{A} \simeq \hat{B}$, como queríamos demonstrar.

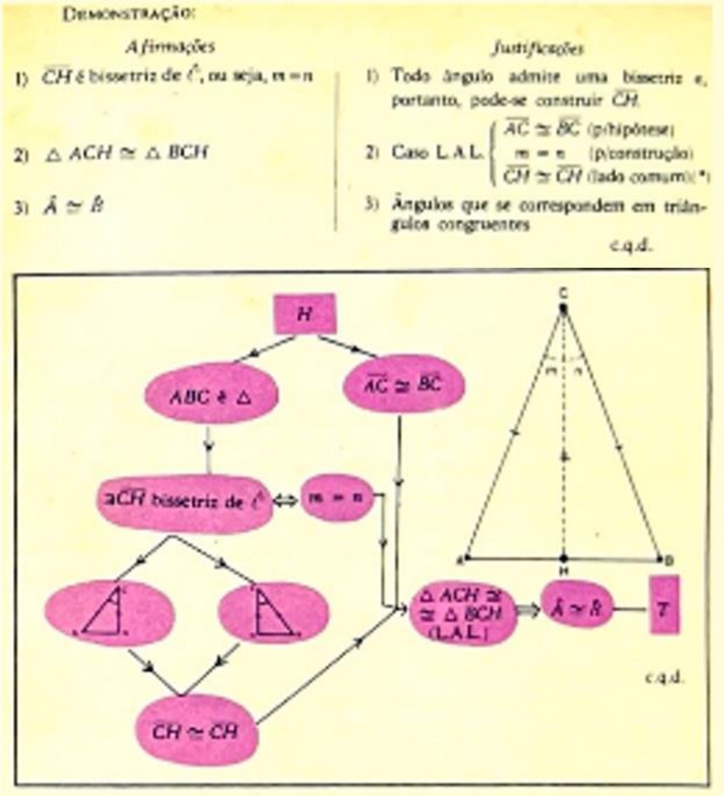
Observação: Se em vez da bissetriz você traçasse a mediana, relativa à base $\overline{AB}$, e seguisse o mesmo raciocínio, provaria também que $\hat{A} \simeq \hat{B}$, porque agora os triângulos obtidos seriam congruentes pelo caso L.L.L. Experimente!

Outro "caminho" para provar que $\hat{A} \simeq \hat{B}$ é traçar a altura, relativa à base $\overline{AB}$, pois nesse caso os triângulos obtidos seriam congruentes por serem retângulos que possuem a hipotenusa e um cateto, respectivamente, congruentes. Sabe por quê?

Porque é sempre possível "compor" um triângulo *isósceles* com dois triângulos retângulos que possuam a hipotenusa e um cateto, respectivamente, congruentes. E tais triângulos são *congruentes*, nessa "composição", pelo caso L.A.A.

Note. From *Matemática: Curso moderno (3º volume para os ginásios)* (Sangiorgi, 1967, p. 240).

Figure 3 – Two-column model and drawn diagrams



Note. From *Matemática: Curso moderno (3º volume para os ginásios)* (Sangiorgi, 1967, p. 241).

Based on the study conducted on representative textbooks from the 1930s and 1950s (Table 1), in which we did not identify the two-column test model, it is plausible to assume that Sangiorgi's textbook for the 3rd year of secondary school was one of the first to incorporate this model into Brazilian textbooks. The fact that the author completed an internship in the United States in 1960, where he had direct contact with widely used teaching materials in that country, reinforces the hypothesis that the inclusion of the two-column test model in the modern collection stems from this international experience— even though the author does not explicitly refer to this influence in his work.

It is also possible to recognize the relevance attributed by Sangiorgi to the different processes of justifying and arguing, bringing together not only proposals linked to the American context, but also European ideas, such as those of the French educator Lucienne Félix (drawn diagrams). The methodological aspect valued in Sangiorgi's modern proposal for teaching geometry becomes evident, as it allows for close dialogue with the reflections and observations reported by teachers and debated in congresses, especially regarding pedagogical practices considered problematic, such as the memorization of proofs by students.

The second textbook in Table 2, *Modern Geometry*, by Edwin E. Moise (Harvard University) and Floyd L. Downs Jr. (Hillsdale College), also circulated in Brazil, and it was intended for students from the 3rd year of middle school to the 3rd year of high school. The work, originally produced in the USA, was translated into Portuguese by Renata Watanabe[16]and Dorival Mello[17], with support from the Publications Department of GEEM, and published by Editora Edgard Blücher in 1971.

A striking innovation of this book was the integration of the study of plane and spatial geometry, unlike the national curricula of 1931, 1942, and 1951, as well as the São Paulo curriculum of 1965, which advocated for their separate approach: plane geometry in middle school (current Elementary School—Final Years) and spatial geometry in high school (current Secondary School). This structural difference would hinder the direct adoption of the work in Brazilian schools, given the current curricular organization.

In the preface to the work, Moise & Downs acknowledge having been strongly influenced by the text *Geometry*, prepared by the Mathematics Study Group[18](SMSG), highlighting that they participated in discussions on mathematics education promoted by the group. The authors argue that: “the student will have a long and varied intuitive experience with geometry in space before beginning a systematic study of this geometry in chapter 8” (Moise & Downs, 1971, Preface), which reinforces the importance of a prior intuitive stage as preparation for deductive study, including proofs and demonstrations.

In the development of the text, before asking students to perform demonstrations, the authors present definitions, concepts, languages, and examples meticulously explained, in addition to already solved demonstrations, presented gradually. As can be seen in the section entitled "Writing Simple Proofs" (Figure 4), the first demonstrations seek to establish a basic argumentative pattern, aiming to introduce students to deductive logic and the rigor characteristic of formal Geometry.

[16] Renata Gompertz Watanabe. She was a professor at Mackenzie University and a member of GEEM.

[17] Dorival Antonio Mello. He was a professor at Mackenzie University and a member of GEEM.

[18] Concern about the increasing need for mathematicians resulted, in 1958, in two conferences sponsored by the *National Science Foundation* (NSF), in which a project was requested to reformulate the school mathematics curriculum and develop textbooks. Edgar Begle was nominated and accepted the position of director of the resulting project *The School Mathematics Study Group* (SMSG). More details can be found in Oliveira Filho (2009).

Figura 4 – Simple demonstrations

4-6. ESCREVENDO DEMONSTRAÇÕES SIMPLES

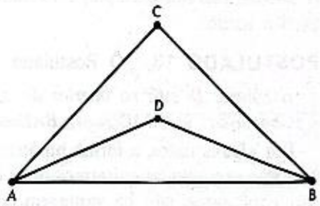
Logo mais, escrever suas próprias demonstrações constituirá uma grande parte de seus exercícios. Será melhor adquirir mais um pouco de prática em escrever demonstrações fáceis antes de atacar as mais difíceis do próximo capítulo. Provàvelmente, a melhor maneira de indicar como devem ser suas demonstrações, é dar mais alguns exemplos. Nesses exemplos e problemas você pode supor que as figuras são coplanares salvo aviso em contrário.

Exemplo 1

Dados: ΔABC e ΔABD, como na figura à direita, com $\angle DAB \cong \angle DBA$ e $\angle CAD \cong$ $\cong \angle CBD$,

Demonstre:

$\angle CAB \cong \angle CBA$



Demonstração

Afirmações	*Justificações*
1. $m\angle DAB = m\angle DBA$.	Dado.
2. $m\angle CAD = m\angle CBD$.	Dado.
3. $m\angle DAB + m\angle CAD$. $= m\angle DBA + m\angle CBD$.	Propriedade aditiva da igualdade.
4. $m\angle CAB = m\angle CBA$.	Postulado da Adição de Ângulos.
5. $\angle CAB \cong \angle CBA$.	Definição de congruência de ângulos.

Note. From *Geometria moderna – Parte I* (Moise & Downs, 1971, p. 88).

As observed in Figure 4, the two-column proof model is employed from Example 1 onwards and remains throughout the book as the standard format for presenting proofs. Next, the authors introduce a specific moment for students to begin writing their own proofs, in Section 5.4—*Inventing Your Own Proofs*. At this stage, Moise and Downs highlight that students should have already acquired the necessary knowledge to produce deductive arguments and emphasize that: “writing your own proofs will be a very important part of your work, and chances are it will be much more fun than reading other people’s proofs” (Moise & Downs, 1971, p. 112).

This movement to encourage students to produce proofs is also evident in Sangiorgi's book for the 3rd year of secondary school (1967), in which the author includes the "friendly reminder" shown in Figure 5, highlighting the importance of the student taking an active role in constructing deductive reasoning..

Figure 5 – Friendly reminder of encouragement to demonstrate theorems

LEMBRETE AMIGO

Não “decore” demonstração de teorema!

Valorize-se, usando qualquer dos métodos apresentados.

Dê o seu próprio “toque” ao empregar tais métodos e você estará realizando-se em Matemática!

Note. From *Matemática: Curso moderno (3º volume para os ginásios)* (Sangiorgi, 1967, p. 258).

In Moise and Downs' book (1971), we also identified an additional preparatory step for the exercise of demonstrating. The first demonstration proposals directed at students are presented in a partially structured way, requesting the filling in of blanks. This procedure can be observed in problem 1 of Figure 6, which guides the student to complete justifications and logical chains previously indicated by the authors. It is, therefore, a pedagogical device that seeks to support the student's transition from reading and understanding demonstrations to the autonomous construction of deductive arguments.

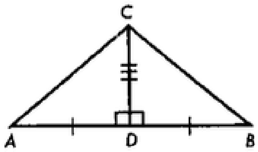
Figure 6 – Initial exercise to demonstrate

1. Copie êste problema em seu caderno e complete as informações que faltam.

Dados: A figura ao lado com

$\overline{CD} \perp \overline{AB}$ e $\overline{AD} \cong \overline{BD}$.

Demostre que: $\Delta ADC \cong \Delta BDC$.



Demonstração

Afirmações	*Justificações*
1. $\overline{AD} \cong \overline{BD}$.	Dado.
2. $\overline{CD} \perp \overline{AB}$.	....
3. $\angle ADC \cong \angle BDC$.	Definições de perpendiculares e ângulo reto.
4. $\overline{CD} \cong \overline{CD}$.	Identidade. (Todo segmento é congruente a si próprio.)
5. $\Delta ADC \cong$	....

Note. From *Geometria moderna – Parte I* (Moise & Downs, 1971, p. 114).

After becoming familiar with the two-column proof model through structured exercises, in which the student completes previously suggested justifications or statements, the problems begin to directly request proof, without the initial support. This intermediate stage—in which the student fills in parts of a guided proof—was not identified in Sangiorgi's book, which highlights the author's options and choices in his appropriation process. He chose to present different forms of representation, instead of exploring the two-column proof method in more detail, as proposed by Moise & Down (1971). This choice reinforces that every appropriation process involves decisions, particularities, and consideration of the context in which it will be implemented. In any case, the method, already consolidated in the USA in the mid-20th century, arrives in the Brazilian school culture embedded in a broader movement, such as the MMM, even though it was not originally conceived as a modernizing proposal for the United States in the 1960s.

In summary, in the book *Modern Geometry*, Moise and Downs (1971) predominantly employ the two-column proof model: we identified approximately 30 occurrences in Part I of the work alone. This fact offers clear evidence for the thesis presented by Herbst (2002), according to which the model had become established as a widely used standard in geometry teaching in American schools, lasting for a long period and becoming a true icon of demonstrative school practice. In the context of the book, the model is treated as a prototypical form of presenting proofs, dispensing further justification for its use—as if the two-column proof were synonymous with the very idea of geometric proof in the American context.

Final remarks

This article aimed to present the creation and consolidation of a demonstration method—the two-columns proof method—as a didactic strategy so that proofs and demonstrations would be conceived as skills learned within the school context. By recovering their emergence in specific contexts, examining their international circulation, and analyzing their first processes of appropriation in Brazil, particularly in São Paulo, we intend to contribute to the historical understanding of demonstrative practices in school geometry, relating them to the processes of argumentation or validation in geometry.

A first reflection concerns the transition from a practice originally confined to academic culture—aimed at mathematicians and geometers, who operated under rigid logical-deductive norms—to an object of school teaching intended for students from the age of 13. On the one hand, we have the universe of formal proof, in which rigor is a structuring condition; on the other, the school context, in which mathematical language, argumentation tools, and logical procedures are in formation and construction. In Brazil, between the 1930s and 1950s, although the regulations insisted on the need for a preparatory stage, such guidelines did not manage to substantially modify the predominance of deductive geometry based on the memorization of proofs. On the other hand, the inclusion of alternative methods for justification and argumentation—among them the two-column proof—may indicate a break with practices considered traditional, recognizing the school as a differentiated locus and, therefore, as a context that requires its own methods, distinct from those traditionally used by mathematicians.

A second reflection concerns the reasons why the two-column proof method was only incorporated in Brazil within the context of the Modern Mathematics Movement (MMM). At that time, the central debate was not limited to the format of the proofs, but to the very axiomatic basis of Geometry—whether it should remain anchored in Euclid, even if revisited, or be reorganized in light of Geometric Transformations. Even so, the two-column proof method operated as a pedagogical resource to highlight logical reasoning. of the arguments, contributing to making the exercise of demonstration more accessible to students.

Finally, we highlight the importance of understanding the circulation and appropriation of knowledge from a transnational perspective. The adoption or adaptation of pedagogical models involves specific times, mediations, exchanges, and cultural agents—teachers, textbook authors, translators, and teacher training institutions. The case of the translation and circulation of Moise & Downs's (1971) *Modern Geometry* is exemplary: although its integrated organization of plane and spatial geometry made its adoption in Brazilian secondary education difficult, the work found a place in the training of mathematics teachers, opening new avenues for the incorporation of demonstrative practices and geometric representations.

The project *Validation Processes in Geometry Contexts: Histories and Connections with Teacher Training* is currently under development and, in one of its sub-projects, aims to investigate the subsequent trajectory of the two-column proof method in Brazil: its diffusion in textbook collections, its permanence or transformation, as well as its relationship with initial and continuing teacher training. We hoped that this will contribute to the history of school geometry and, in particular, to the understanding of the ways in which demonstrative practices are appropriated, reinterpreted, and taught at different levels of schooling.

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